Titre

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Local Theory of the Cauchy problem for NLW ($N = 3$)

We first review a few facts about the linear wave equation:

$$\text{(LW)} \qquad \begin{cases} (\partial_t^2 - \Delta)w = h(x,t), & (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ w\,|_{t=0} = w_0 \\ \partial_t\, w\,|_{t=0} = w_1 \end{cases}$$

Fourier method:

$$w(t) = \cos(\sqrt{-\Delta}\,t)w_0 + \frac{\sin(\sqrt{-\Delta}\,t)}{\sqrt{-\Delta}}w_1 + \int_0^t \frac{\sin(t-t')\sqrt{-\Delta}}{\sqrt{-\Delta}}\,h(t')dt',$$

$$w(t) = S_L(t)(w_0, w_1) + D_L(t)(h).$$

Here,

$$(\cos(\sqrt{-\Delta}\,t)f)^{\wedge}(\xi) = \cos(|\xi|t)\hat{f}(\xi), \left(\frac{1}{\sqrt{-\Delta}}f\right)^{\wedge}(\xi) = \frac{\hat{f}(\xi)}{|\xi|}, \text{ etc.}$$

One of the main properties is finite speed of propagation:

If $\operatorname{supp}(w_0, w_1) \cap \overline{B(x_0, a)} = \phi$, and

$$\operatorname{supp} h \cap \left(\bigcup_{0 \le t \le a} B(x_0, a-t) \times \{t\} \right) = \phi,$$

then $w \equiv 0$ on $\bigcup\limits_{0 \le t \le a} B(x_0, a-t) \times \{t\}$.

In odd dimensions: strong Huygens principle; if $h \equiv 0$, $\operatorname{supp}(w_0, w_1) \subset B(x_0, a)$, then for $t > 0$, $\operatorname{supp} \vec{w} \subset \{x : t - a \le |x - x_0| \le a + t\}$ (in even dims, only upper bound), where $\vec{w} = (w, \partial_t w)$.

In order to study the local theory of the Cauchy problem for (NLW):

$$\text{(NLW)} \qquad \begin{cases} \partial_t^2 u - \Delta u = u^5, & x \in \mathbb{R}^3, t \in \mathbb{R} \\ u\,|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^3) \\ \partial_t\, u\,|_{t=0} = u_1 \in L^2(\mathbb{R}^3) \end{cases}$$

We recall the important Strichartz estimates:

$$\text{(S1)} \qquad \sup_t \|(w(t), \partial_t\, w(t))\|_{\dot{H}^1 \times L^2} + \|w\|_{L^8_{tx}} + \|w\|_{L^5_t L^{10}_x} +$$

$$+ \|D^{-1/2}\, \partial_t\, w\|_{L^4_{tx}} + \|D^{1/2} w\|_{L^4_{tx}} \leq C \left\{ \|(w_0, w_1)\|_{\dot{H}^1 \times L^2} + \|D^{1/2} h\|_{L^{4/3}_{tx}} \right\},$$

$$(S2) \qquad \sup_t \|(w(t), \partial_t w(t))\|_{\dot H^1\times L^2} + \|w\|_{L^8_{tx}} + \|w\|_{L^5_t L^{10}_x} +$$

$$+ \|D^{1/2} w\|_{L^4_{tx}} + \|D^{-1/2}\partial_t w\|_{L^4_{tx}} \le C\left\{\|(w_0, w_1)\|_{\dot H^1\times L^2} + \|h\|_{L^1_t L^2_x}\right\}.$$

Another important technical tool is the Leibniz rule for fractional derivatives. In the case at hand, it says, for an arbitrary interval of time I, that

$$\|D^{1/2}(u^5)\|_{L_I^{4/3} L_x^4} \le C\|u\|^4_{L^8_I L^8_x}\|D^{1/2}u\|_{L^4_I L^4_x},$$

and

$$\|D^{1/2}(u^5 - v^5)\|_{L_I^{4/3}L_x^{4/3}} \leq C\left[\|u^4\|_{L_I^2 L_x^2} + \|v^4\|_{L_I^2 L_x^2}\right]$$

$$\cdot \quad \|D^{1/2}(u - v)\|_{L_I^4 L_x^4} \leq C\left[\|u^3\|_{L_I^{8/3}L_x^{8/3}} + \|v^3\|_{L_I^{8/3}L_x^{8/3}}\right]$$

$$\cdot \quad \left[\|D^{1/2}u\|_{L_I^4 L_x^4} + \|D^{1/2}v\|_{L_I^4 L_x^4}\right]\|u - v\|_{L_I^8 L_x^8}$$

We now introduce the notation, for I a time interval:

$$S(I) = L_I^8 L_x^8, \quad W(I) = L_I^4 L_x^4$$

A solution of (NLW) on an interval I, where $0 \in I$, is a function $u \in C(\overset{\circ}{I}; \dot{H}^1)$ such that $\partial_t u \in C(\overset{\circ}{I}; L^2)$ and $\forall J \subset\subset I$,

$$\|u\|_{S(J)} + \|D^{1/2} u\|_{W(J)} + \|D^{-1/2}\,\partial_t u\|_{W(J)} < \infty,$$

and u satisfies the Duhamel formulation

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t \frac{\sin(t - t')\sqrt{-\Delta}}{\sqrt{-\Delta}} u^5(t')dt'.$$

The main result on the local Cauchy problem is that for any initial condition $(u_0, u_1) \in \dot{H}^1 \times L^2$, there is a unique solution u defined on a maximal interval of definition $I\max(u) = (T_-(u), T_+(u))$. To show this, one uses the contraction mapping principle, combined with the chain rule, to show that $\exists\, \delta_0 > 0$ such that, if $\|S_L(\cdot)(u_0, u_1)\|_{S(I)} = \delta < \delta_0$, then $\exists\,!u$, a solution on I, with initial condition (u_0, u_1). Moreover, if $A = \|D^{1/2} S_L(\cdot)(u_0, u_1)\|_{W(I)}$, we have that

$$\|u(\cdot) - S_L(\cdot)(u_0, u_1)\|_{S(I)} + \sup_{t \in I} \|\vec{u}(t) - \vec{S}_L(t)(u_0, u_1)\|_{\dot{H}^1 \times L^2} +$$

$$+ \|u(\cdot) - S_L(\cdot)(u_0, u_1)\|_{L^5_I L^{10}_x} \leq CA\delta^4.$$

One also uses the fact that if $u^{(1)}, u^{(2)}$ are solutions of (NLW) on I', with $(u^{(1)}(0), \partial_t u^{(1)}(0)) = (u^{(2)}(0), \partial_t u^{(2)}(0))$, then $\vec{u}^{(1)} = \vec{u}^{(2)}$ on I. This follows from the uniqueness of fixed points and an iteration argument.

Notice that 0 can be replaced by any t_0.

Finite time blow-up/scattering criterion

If $T_+(u) < \infty$, then $\|u\|_{S(0,T_+(u))} = \infty$. This is again shown by an "iteration". Hence, if $\|u\|_{S(0,T_+(u))} < \infty$, then $T_+(u) = \infty$. Moreover, in this case the solution scatters forward in $\dot{H}^1 \times L^2$, i.e. $\exists$ a solution v_L of (LW) such that $\lim_{t\to\infty} \|\vec{u}(t) - \vec{v}_L(t)\|_{\dot{H}^1\times L^2} = 0$. Of course analogous statements hold backward in time.

Remark: Since the existence result is obtained by the contraction mapping principle, the solution can be locally in time constructed by Picard iteration. Hence, finite speed of propagation also holds for (NLW). For instance, if $\text{supp}(u_0, u_1) \cap \overline{B(x_0, a)} = \phi$, then $\text{supp}(u(t), \partial_t u(t)) \subset (\mathbb{R}^3 \times I) \setminus \bigcup_{0\le t\le a} \overline{B(x_0, a-t)} \times \{t\}$. A similar remark holds for two solutions agreeing initially on $\overline{B(x_0, a)}$. Note that the strong Huygens principle may not apply.

There are other equivalent notions of solution. For instance assume that $0 \in I$, $u \in C(\overset{\circ}{I}; \dot{H}^1 \times L^2)$, that $u \in L^5_J L^{10}_x$, $\forall J \subset\subset I$ and that u verifies the integral equation. Then from (S2) it is easy to see that u is a solution of (NLW) in I. In fact, we have more:

Proposition: Let $(u_0, u_1) \in \dot{H}^1 \times L^2$, $0 \in \overset{\circ}{I}$, $u \in L^5_J L^{10}_x$, $\forall J \subset\subset I$, $u \in C(\overset{\circ}{I}; \dot{H}^1 \times L^2)$, $(u(0), \partial_t u(0)) = (u_0, u_1)$ and $\partial_t^2 u - \Delta u = u^5$ in $\mathcal{D}'(\mathbb{R}^3 \times I)$. Then u solves (NLW).

This is useful because it allows us, in some circumstances, not to use fractional derivatives.

An important consequence of the "local Cauchy theory" is a long-time perturbation result.

Proposition: Let $0 \in I$. Let $\vec{U} \in C(I; \dot{H}^1 \times L^2)$ be such that $\|U\|_{L^5_I L^{10}_x} \le M$ and be a solution to

$$\partial_t^2 U - \Delta U = U^5 + f \qquad \text{in} \quad \mathbb{R}^3 \times I.$$

Then $\exists\, \varepsilon_* = \varepsilon_*(M)$ sufficiently small, such that if $\|f\|_{L^1_I L^2_x} + \|(u_0, u_1) - \vec{U}(0)\|_{\dot{H}^1 \times L^2} \le \varepsilon \le \varepsilon^*$, then the unique solution $\vec{u}$ to (NLW) with initial data (u_0, u_1) exists in $\mathbb{R}^3 \times I$. Moreover,

$$\sup_{t \in I} \|\vec{u}(t) - \vec{U}(t)\|_{\dot{H}^1 \times L^2} + \|u - U\|_{L^5_I L^{10}_x} \le C(M)\varepsilon.$$

Remark: There are other versions of this Proposition, using instead of the $L^5_I L^{10}_x$ norm, the $S(I)$ norm, and involving also the $W(I)$ norm.

There are some interesting consequences:

Cor: $K \subset\subset \dot{H}^1 \times L^2$. Then $\exists\, T^+_K > 0$, $T^-_K < 0$ such that $\forall\, (u_0, u_1) \in K$, $T_+(u_0, u_1) \geq T^+_K$, $T_-(u_0, u_1) \leq T^-_K$.

Cor: If $(\tilde{u}_0, \tilde{u}_1) \in \dot{H}^1 \times L^2$ and $(u_{0n}, u_{1n}) \to (\tilde{u}_0, \tilde{u}_1)$ in $\dot{H}^1 \times L^2$, then $T_-(\tilde{u}_0, \tilde{u}_1) \geq \overline{\lim}\, T_-(u_{0n}, u_{1n})$, $T_+(\tilde{u}_0, \tilde{u}_1) \leq \overline{\lim}\, T_+(u_{0n}, u_{1n})$ and $\forall\, t \in (T_-(\tilde{u}_0, \tilde{u}_1), T_+(\tilde{u}_0, \tilde{u}_1))$, $(u_n(t), \partial_t u_n(t)) \to (\tilde{u}(t), \partial_t \tilde{u}(t))$ in $\dot{H}^1 \times L^2$.

Finally, let us discuss the action of Lorentz transformation on solutions, a topic that is very important in the theory. We start with a preliminary lemma for the linear case.

Lemma: Let w be a solution of (LW), $(w_0, w_1) \in \dot{H}^1 \times L^2$, $h \in L^1_t L^2_x$. Then, for $|\ell| < 1$

$$\sup_t \left\| \nabla_{x,t} w \left(\frac{x_1 - \ell t}{\sqrt{1-\ell^2}}, x', \frac{t - \ell x_1}{\sqrt{1-\ell^2}} \right) \right\|_{L^2_{x_1, x'}} \leq$$

$$\leq C \left\{ \|(w_0, w_1)\|_{\dot{H}^1 \times L^2} + \|h\|_{L^1_t L^2_x} \right\}.$$

The proof easily reduces to showing that if $v(x,t) = U(t)f$, with $\hat{v}(\xi,t) = e^{it|\xi|}\hat{f}(\xi)$, then

$$\sup_t \left\| v\left(\frac{x_1-\ell t}{\sqrt{1-\ell^2}}, x', \frac{t-\ell x_1}{\sqrt{1-\ell^2}}\right)\right\|_{L^2(dx_1,dx')} \leq C\|f\|_{L^2}.$$

This is a simple consequence of Plancherel and a change of variables in frequency. From the proof, it is easy to see that $t \mapsto w\left(\frac{x_1-\ell t}{\sqrt{1-\ell^2}}, x', \frac{t-\ell x_1}{\sqrt{1-\ell^2}}\right)$ is continuous into $\dot{H}^1 \times L^2$.

For $\ell \in (-1,1)$, define for $(y,s) \in \mathbb{R}^3 \times \mathbb{R}$,

$$(x,t) = \Phi_\ell(y,s) = \left(\frac{y_1+\ell s}{\sqrt{1-\ell^2}}, y', \frac{s+\ell y_1}{\sqrt{1-y^2}}\right),$$

where $x' = (x_2,x_3)$, $y' = (y_2,y_3)$. Thus $(y,s) = \Phi_\ell^{-1}(x,t) = \left(\frac{x_1-\ell t}{\sqrt{1-\ell^2}}, x', \frac{t-\ell x_1}{\sqrt{1-\ell^2}}\right)$.

Let u be a globally defined solution u of (NLW), we define its Lorentz transform u_ℓ by:

$$u_\ell(x,t) = u\left(\frac{x_1 - \ell t}{\sqrt{1-\ell^2}}, x', \frac{t - \ell x_1}{\sqrt{1-\ell^2}}\right) = u(\Phi_\ell^{-1}(x,t))$$

since u is global in time, u_ℓ is well defined as an element of $L^8_{\text{loc}}(\mathbb{R}^4)$. (This fact led KM to the notion of solution based on L^8 that we have developed here.) We have:

Proposition: Let u be a global, finite energy solution of (NLW). Then, u_ℓ is also a global, finite energy solution of (NLW).

Profile Decomposition:
This is a fundamental tool in the implementation of the concentration-compactness method to evolution problems. It can be thought of as a way to quantify the lack of compactness in the embedding given by (S1) and (S2).

Definition: Let $\{(u_{0n}, u_{1n})\}_n$ be a bounded sequence in $\dot{H}^1 \times L^2$. For $j \geq 1$, let U_L^j be a solution of (LW) and let $\{\lambda_{jn}; x_{jn}; t_{jn}\}_n$ be a sequence of parameters in $(0,\infty) \times \mathbb{R}^3 \times \mathbb{R}$. The sequence of parameters are said to be orthogonal if for all $j \geq 1$,

$$j \neq k \Rightarrow \lim_{n\to\infty} \frac{\lambda_{jn}}{\lambda_{kn}} + \frac{\lambda_{kn}}{\lambda_{jn}} + \frac{|t_{jn} - t_{kn}|}{\lambda_{jn}} + \frac{|x_{jn} - x_{kn}|}{\lambda_{jn}} = \infty.$$

We say that $(U^j_L, \{\lambda_{jn}, x_{jn}, t_{jn}\}_n)_{j\geq 1}$ is a profile decomposition of the sequence $\{(u_{0n}, u_{1n})\}_n$ if the parameters are orthogonal, and denoting by

and
$$U^j_{Ln}(x,t) = \frac{1}{\lambda_{jn}^{1/2}} U^j_L\left(\frac{x - x_{jn}}{\lambda_{jn}}, \frac{t - t_{jn}}{\lambda_{jn}}\right),$$
$$w^J_n(x,t) = S_L(t)(u_{0n}, u_{1n}) - \sum_{j=1}^{J} U^j_{Ln}(x,t),$$

we have $\overline{\lim_{J\to\infty}}\ \overline{\lim_n} \|(w^J_{0n}, w^J_{1n})\|_{\dot{H}^1\times L^2} < \infty$, where $(w^J_{0n}, w^J_{1n}) = \vec{w}^J_n(x,0)$ and

$$\lim_{J\to\infty} \overline{\lim_n} \|w^J_n\|_{S(\mathbb{R})} = 0$$

Bahouri-Gérard ($N = 3$, Bulut $N > 3$) proved that for any bounded sequence in $\dot{H}^1 \times L^2$, a subsequence always admits a profile decomposition, and the error w^J_n in addition verifies $\overline{\lim_{J\to\infty}}\ \overline{\lim_n} \|w^J_n\|_{L^5_t L^{10}_x} + \|w^J_n\|_{L^\infty_t L^6_x} = 0$ (but not necessarily $\|(w^J_{0n}, w^J_{1n})\|_{\dot{H}^1\times L^2} \to 0$).

Remark: The profiles U_L^j are constructed as weak limits.

For each j,

$$\vec{S}_L(t_{jn}/\lambda_{jn})\left(\lambda_{jn}^{1/2}u_{0n}(\lambda_{jn}\cdot+x_{jn}),\lambda_{jn}^{3/2}u_{1n}(\lambda_{jn}\cdot+x_{jn})\right)$$

$\underset{n}{\rightharpoonup}$ $\vec{U}_L^j(0)$ weakly in $\dot{H}^1\times L^2$.

This property is in fact a consequence of the orthogonality of the parameters and the property:

$$j\leq J\Rightarrow\left(\lambda_{jn}^{1/2}w_n^J(\lambda_{jn}\cdot+x_{jn},t_{jn}),\lambda_{jn}^{3/2}\partial_t w_n^J(\lambda_{jn}\cdot+x_{jn},t_{jn})\right)$$

$\underset{n}{\rightharpoonup}(0,0)$ weakly in $\dot{H}^1\times L^2$. One can in fact show that this last property holds for any profile decomposition.

The following Pythagorean expansions are important for the applications: (see Bahouri-Gérard)

$$\lim_{n\to\infty}\Bigg[\|u_{0n}\|^2_{\dot H^1}+|u_{1n}\|^2-\Bigg(\sum_{j=1}^{J}\|U^j_{Ln}(0)\|^2_{\dot H^1}+\|\partial_t U^j_{Ln}(0)\|^2_{L^2}+$$

$$+\|w^J_{0n}\|^2_{\dot H^1}+\|w^J_{1n}\|^2_{L^2}\Bigg)\Bigg]=0$$

and

$$\lim_{n\to\infty}\left[\|u_{0n}\|^6_{L^6}-\left(\sum_{j=1}^{J}\|U^j_{Ln}(0)\|^6_{L^6}+\|w^J_{0n}\|^6_{L^6}\right)\right]=0$$

for all $J\geq 1$.

By extracting subsequences and possibly changing the profiles by t translates (in the case when $|t_{jn}|/\lambda_{jn}$ is bounded), we can always assume that one of the following holds: $t_{jn} \equiv 0$ or $t_{jn}/\lambda_{jn} \to +\infty$ or $t_{jn}/\lambda_{jn} \to -\infty$. In the first case we say that j is core, in the second and third that j is scattering.

Remark: It is tempting to assume that the following Pythagorean expansions also hold, for $J \geq 1$.

$$\|u_{0n}\|_{\dot{H}^1}^2 = \sum_{j=1}^{J} \|U_{L,n}^j(0)\|_{\dot{H}^1}^2 + \|w_{0n}^J\|_{\dot{H}^1}^2 + o_n(1)$$

$$\|u_{1n}\|_{L^2}^2 = \sum_{j=1}^{J} \|\partial_t\, U_{L,n}^j(0)\|_{L^2}^2 + \|w_{1n}^J\|^2 + o_n(1).$$

In fact, surprisingly, they fail as can be seen by counterexample. However, a version of them still holds and this will be useful to us.

Lemma: ([DKM]) After extraction in n, we have that, for any given $\varepsilon_0 > 0$, $\exists\,\overline{J} = \overline{J}(\varepsilon_0, A)$, where A is a uniform bound for the $\dot{H}^1 \times L^2$ norm of (u_{0n}, u_{1n}), and for each $J \geq \overline{J}$ there exists $\bar{n} = \bar{n}(\varepsilon_0, A, J)$ such that, if $n \geq \bar{n}$ we have

$$\|u_{0n}\|_{\dot{H}^1}^2 = \sum_{\substack{j=1\\ j\in\mathcal{J}}}^{J} \|U_L^j(0)\|_{\dot{H}^1}^2 + \left\|\sum_{\substack{j=1\\ j\notin\mathcal{J}}}^{J} U_{L,n}^j(0)\right\|_{\dot{H}^1}^2 + \|w_n^J(0)\|_{\dot{H}^1}^2 + \varepsilon_{n,J}$$

$$\|u_{1n}\|_{L^2}^2 = \sum_{\substack{j=1\\ j\in\mathcal{J}}}^{J} \|\partial_t U_L^j(0)\|_{L^2}^2 + \left\|\sum_{\substack{j=1\\ j\notin\mathcal{J}}}^{J} \partial_t U_{L,n}^j(0)\right\|_{L}^2 + \|\partial_t w_n^J(0)\|_{L^2}^2 + \varepsilon_{n,J},$$

where $|\varepsilon_{n,J}| \leq \varepsilon_0$ and $\mathcal{J}$ denotes the set of core indices j.

At this point, one naturally wonders about the issue of uniqueness (or lack of) of the profile decomposition. This topic was clarified by DKM.

Lemma: Let $\left(U_L^j, \{\lambda_{jn}, x_{jn}, t_{jn}\}_n\right)_j$ be a profile decomposition of the bounded sequence in $\dot{H}^1 \times L^2 \{(u_{0n}, u_{1n})\}_n$. For all $j \geq 1$, consider sequences $\{\tilde{\lambda}_{jn}, \tilde{x}_{jn}, \tilde{t}_{jn}\}$ in $(0,0) \times \mathbb{R}^3 \times \mathbb{R}$ such that for all $j \geq 1$ there exists $(\mu_j, y_j, s_j) \in (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\tilde{\lambda}_{jn}}{\lambda_{jn}} = \mu_j \quad \lim_{n\to\infty} \frac{\tilde{x}_{jn} - x_{jn}}{\lambda_{jn}} = y_j, \quad \lim_{n\to\infty} \frac{\tilde{t}_{jn} - t_{jn}}{\lambda_{jn}} = s_j.$$

Let $\tilde{U}_L^j(x,t) = \mu_j^{1/2} U_L^j(y_j + \mu_j x, s_j + \mu_j t)$. Then, $\left(\tilde{U}_L^j, \{\tilde{\lambda}_{jn}, \tilde{x}_{jn}, \tilde{t}_{jn}\}_n\right)_j$ is also a profile decomposition of the sequence $\{(u_{0n}, u_{1n})\}_n$.

Moreover, we have the following uniqueness result:

Lemma: Let $\left(U_L^j, \{\lambda_{jn}, x_{jn}, t_{jn}\}_n\right)_j$ and $\left(\tilde{U}_L^j, \{\tilde{\lambda}_{jn}, \tilde{x}_{jn}, \tilde{t}_{jn}\}_n\right)_j$ be two profile decompositions of the same bounded sequence $\{(u_{0n}, u_{1n})\}_n$. Assume that each of the sets

$$\mathcal{J} = \{j \geq 1 : U_L^j \neq 0\}, \quad \mathcal{K} = \{k \geq 1 : \tilde{U}_L^k \neq 0\},$$

is finite, or equal to $\mathbb{N}\backslash\{0\}$. Then, after extraction in n, there exists a unique one to one map $\varphi : \mathbb{N}\backslash\{0\} \to \mathbb{N}\backslash\{0\}$, with the following property. For all $j \geq 1$, letting $k = \varphi(j)$, then $U_L^j = 0$ if and only if $\tilde{U}_L^k = 0$. Furthermore, if $U_L^j \neq 0$, there exists $(\mu_j, y_j, s_j) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}$ such that

$$\lim_n \frac{\tilde{\lambda}_{kn}}{\lambda_{jn}} = \mu_j \quad \lim_n \frac{\tilde{x}_{kn} - x_{jn}}{\lambda_{jn}} = y_j \quad \lim_n \frac{\tilde{t}_{kn} - t_{jn}}{\lambda_{jn}} = s_j$$

and $\tilde{U}_L^k(x,t) = \mu_j^{1/2} U_L^j(y_j + \mu_j x, s_j + \mu_j t)$.

Remark: It is easy to see that if for a sequence $\{(\lambda_n, x_n, t_n)\}_n$ in $(0,\infty) \times \mathbb{R}^3 \times \mathbb{R}$, $\vec{S}_L(t_n/\lambda_n)\,(\lambda_n^{1/2} u_{0n}(\lambda_n \cdot + x_n), \lambda_n^{3/2} u_{1n}(\lambda_n \cdot + x_n)) \underset{n}{\rightharpoonup} (v_0, v_1)$ weakly in $\dot{H}^1 \times L^2$, then, modulo a transformation as in the previous Lemmas, $(v_0, v_1) = \vec{U}_L^j(0)$ for a profile U_L^j in the profile decomposition of $\{(u_{0n}, u_{1n})\}_n$, with parameters (λ_n, x_n, t_n).

In order to apply the profile decomposition to nonlinear problems, we need to introduce the notion of nonlinear profiles.

Definition: Let $j \geq 1$. A nonlinear profile U^j associated to the linear profile U_L^j and the sequence of parameters $\{\lambda_{jn}, t_{jn}\}$ is a solution U^j of (NLW) such that, for large n, $-t_{jn}/\lambda_{jn} \in I\max(U^j)$ and

$$\lim_{n\to\infty} \left\| \vec{U}_L^j(-t_{jn}/\lambda_{jn}) - \vec{U}^j(-t_{jn}/\lambda_{jn}) \right\|_{\dot{H}^1 \times L^2} = 0.$$

After extraction, we can always assume that for all $j \geq 1$, the following limit exists: $\lim_n -t_{jn}/\lambda_{jn} = \sigma_j \in [-\infty, +\infty]$.

Using the local theory of the Cauchy problem for (NLW) when $\sigma_j \in \mathbb{R}$, and the existence of the wave operators for $\sigma_j \in \{-\infty, \infty\}$, one can see that for each j, there exists a unique nonlinear profile U^j associated to U^j_L and $\{\lambda_{jn}, t_{jn}\}_n$. If $\sigma \in \mathbb{R}$, then $\sigma_j \in (T_-(U^j), T_+(U^j))$. If $\sigma_j = -\infty$, then $T_-(U^j) = -\infty$ and U^j scatters at $-\infty$. If $\sigma_j = +\infty$, then $T_+(U^j) = +\infty$ and U^j scatters forward in time.

Let $U^j_n(x,t) = \frac{1}{\lambda_{jn}^{1/2}} U^j\left(\frac{x-x_{jn}}{\lambda_{jn}}, \frac{t-t_{jn}}{\lambda_{jn}}\right)$, a solution of (NLW). The maximal positive time of existence for U^i_n is $\lambda_{jn} T_+(U^j) + t_{jn}$ (or $+\infty$ if $T_+(U^j) = +\infty$). The nonlinear profiles are used as "building blocks" of solutions, through the following approximation theorem, which is a fundamental tool for us.

Theorem: Let $\{(u_{0n}, u_{1n}\}_n$ be a bounded sequence in $\dot{H}^1 \times L^2$ which admits a profile decomposition. Let u_n be the solution of (NLW) with initial data (u_{0n}, u_{1n}).

i) Assume that for all j, U^j scatters forward in time. Then, for large n, u_n scatters forward in time and letting $r_n^J(t) = u_n - \sum_{j=1}^{J} U_n^j(t) - w_n^J(t)$, $t \geq 0$, we have

$$\lim_{J\to\infty}\left[\overline{\lim_n}\left(\|r_n^J\|_{S(0,\infty)} + \sup_{t\in[0,\infty)} \|r_n^J(t)\|_{\dot{H}^1\times L^2}\right)\right] = 0.$$

ii) Let $\theta_n \in (0,\infty)$. Assume that for all j, n

$$\frac{\theta_n - t_{jn}}{\lambda_{jn}} < T_+(U^j) \quad \text{and} \quad \overline{\lim_n}\, \|U^j\|_{S\left(\frac{-t_{jn}}{\lambda_{jn}}, \frac{\theta_n - t_{jn}}{\lambda_{jn}}\right)} < \infty.$$

Then for large n, u_n is defined on $[0,\theta_n)$ and $\overline{\lim_n}\, \|u_n\|_{S(0,\theta_n)} < \infty$.

Moreover, for $t \in [0, \theta_n)$, we have

$$\vec{u}_n(x,t) = \sum_{j=1}^{J} \vec{U}_n^j(x,t) + w_n^J(x,t) + r_n^J(x,t),$$

where

$$\lim_{J\to\infty}\left[\overline{\lim_{n}}\left(\|r_n^J\|_{S(0,\theta_n)} + \sup_{t\in[0,\theta_n)} \|\vec{r}_n^{\,J}(t)\|_{\dot{H}^1\times L^2}\right)\right] = 0.$$

Remark: Assume that for each j, one of

a) $\|(U^j(0), \partial_t U^j(0)\|_{\dot{H}^1\times L^2} \leq \eta_0$

b) $\lim_{n\to\infty} -\frac{t_{jn}}{\lambda_{jn}} = +\infty$

c) $\overline{\lim_{n}} \frac{\theta_n - t_{jn}}{\lambda_{jn}} < T_+(U^j)$

Then the hypothesis of ii) above hold.

Remark: The S norm can be replaced by the $L_t^5 L_x^{10}$ norm. The proof of the approximation theorem depends crucially on the long time perturbation theorem. The proof uses the following Pythagorean expansion for the S norm of the profiles, which is a consequence of the orthogonality of the parameters:

$$\forall J \geq 1,\ \lim_{n\to\infty} \left[\left\| \sum_{j=1}^{J} U_n^j \right\|_{S(0,\theta_n)}^8 - \sum_{j=1}^{J} \|U_n^j\|_{S(0,\theta_n)}^8 \right] = 0.$$

An important consequence of the approximation theorem is:

Cor: For any $\{\sigma_n\}_n$ such that $0 < \sigma_n < \theta_n$, we have, for fixed J

$$\lim_{n\to\infty}\left[\|\vec{u}_n(\sigma_n)\|^2_{\dot{H}^1\times L^2} - \sum_{j=1}^{J}\left\|U^j\left(\frac{\sigma_n - t_{jn}}{\lambda_{jn}}\right)\right\|^2_{\dot{H}^1\times L^2} - \|\vec{w}_n^J(\sigma_n)\|^2_{\dot{H}^1\times L^2}\right] = 0$$

and

$$\|u_n\|^8_{S(0,\theta_n)} = \sum_{j=1}^{J}\|U_n^j\|^8_{S(0,\theta_n)} + \varepsilon_{n,J}, \quad \text{where} \quad \lim_J \overline{\lim_n}\,|\varepsilon_{nJ}| = 0.$$

This follows because, if $s_{jn} = \frac{\sigma_n - t_{jn}}{\lambda_{jn}} \underset{n}{\longrightarrow} s_j \in [-\infty, +\infty]$ for each j after extraction, and V_L^j is the unique solution of (LW) such that

$$\lim_n \|\vec{V}_L^j(s_{jn}) - \vec{U}^j(s_{jn})\|_{\dot{H}^1 \times L^2},$$

and we let $t'_{jn} = \sigma_n + t_{jn}$, then $(V_L^j, \{\lambda_{jn}, x_{jn}, t'_{jn}\}_n)_j$ is a profile decomposition for $\{\vec{u}_n(\sigma_n)\}$.

Ground state, solutions to the Elliptic Equation, Variational estimates, traveling waves

The elliptic equation associated with (NLW) is, for $Q \in \dot{H}^1$,

$$\text{(EE)} \qquad \Delta Q + Q^5 = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Clearly any solution to the elliptic equation gives rise to a time independent solution of (NLW). We let $\Sigma = \{Q \neq 0,\, Q \text{ a solution of the elliptic equation}\}$. There is an explicit solution, $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2}$, which arose in connection with the Yamabe problem. W is, up to translating and scaling, the only non-negative element of Σ, i.e. $W_{\lambda_0}(x + x_0) = \lambda_0^{-1/2} W\left(\frac{x+x_0}{\lambda_0}\right)$ are the only nonnegative solutions of (EE). W is the only radial solution of (EE), i.e. $\pm W_{\lambda_0}(x)$ are the only radial solutions of (EE). (Gidas-Ni-Nirenberg, Pohozaev, Kwong).

W also has a characterization as the extremal in the

$$\|f\|_{L^6} \leq C_3 \|\nabla f\|_{L^2},$$

and if f verifies $\|f\|_{L^6} = C_3 \|\nabla f\|_{L^2}, f \neq 0$, then $f(x) = \pm W_{\lambda_0}(x + x_0)$.

This is why W is called the “ground state”. (Aubin, Talenti). Note that, by (EE), $\int |\nabla W|^2 = \int W^6$, while $C_3 \|\nabla W\| = \|W\|_{L^6}$, so that $\int |\nabla W|^2 = C_3^{-3}$. Moreover,

$$\begin{aligned} \mathcal{E}(W) = E(W, 0) &= \frac{1}{2}\int |\nabla W|^2 - \frac{1}{6}\int W^6 = \frac{1}{3}\int |\nabla W|^2 \\ &= \frac{1}{3C_3^3} = \frac{1}{3}\int |\nabla W|^2. \end{aligned}$$

Another useful result regarding W, is the following

Theorem: If $Q \in \Sigma$, $\int |\nabla Q|^2 < 2\int |\nabla W|^2$, then $Q(x) = \pm W_{\lambda_0}(x + x_0)$.

Sketch of proof: $Q_+ = \max(Q,0)$, $Q_- = \min(Q,0) = Q - Q_+$. By Kato's inequality, $\Delta Q_+ + Q_+^5 \geq 0$. Hence $\int |\nabla Q_+|^2 \leq \int Q_+^6$. Similarly, $\int |\nabla Q_-|^2 \leq \int Q_-^6$. Since $\int |\nabla Q_+|^2 + \int |\nabla Q_-|^2 = \int |\nabla Q|^2 < 2\int |\nabla W|^2$, $\int |\nabla Q_\pm|^2 < \int |\nabla W|^2$ for at least one choice of $+$ or $-$. Assume, without loss of generality, that it is the $-$. We will show that $Q_- \equiv 0$. Indeed,

$$\int |\nabla Q_-|^2 \leq \int |Q_-|^6 \leq C_3^6 \left(\int |\nabla Q_-|^2\right)^3 = \left(\int |\nabla W|^2\right)^{-2} \cdot \left(\int |\nabla Q_-|^2\right)^3.$$

Thus, $Q_- \equiv 0$, or $\int |\nabla W|^2 \leq \int |\nabla Q_-|^2$, which contradicts our assumption. Hence $Q \geq 0$ is in Σ and thus has the desired form.

Corollary: If $Q \in \Sigma$, $\mathcal{E}(Q) \leq \mathcal{E}(W)$, then $Q = \pm W_{\lambda_0}(x + x_0)$. This follows because $\mathcal{E}(Q) = \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{6}\int Q^6 = \frac{1}{3}\int |\nabla Q|^2$ by (EE). Hence $\int |\nabla Q|^2 \leq \int |\nabla W|^2$ by our assumption, and hence by the Theorem, Q has the claimed form.

We next turn to further variational estimates which follow from the properties of the ground-state W, and which will be used later for the ground-state conjecture.

Lemma: Assume that $\|\nabla v\|^2 < \|\nabla W\|^2$ and $\mathcal{E}(v) = E(v,0) \leq (1 - \delta_0)\, E(W,0)$, for some $0 < \delta_0 < 1$. Then, $\exists\, \bar{\delta} = \bar{\delta}(\delta_0)$ such that

$$\|\nabla v\|^2 \leq (1 - \bar{\delta})\|\nabla W\|^2$$

$$\int |\nabla v|^2 - v^6 \geq \bar{\delta} \int |\nabla v|^2$$

Sketch of proof: Let $f(y) = \frac{1}{2}y - \frac{1}{6}C_3^6 y^3$. Note that if $\bar{y} = \|\nabla v\|^2$, then $f(\bar{y}) \leq \mathcal{E}(v) = E(v,0)$. Also note that $f(y) = 0 \Leftrightarrow y = 0$ or $y = y^* = 3^{1/2}\frac{1}{C_3^3} = 3^{1/2}\|\nabla W\|^2$. Note also that for $y > 0, f'(y) = 0 \Leftrightarrow y = y_c = \frac{1}{C_3^3} = \|\nabla W\|^2, f(y_c) = \frac{1}{3C_3} = \mathcal{E}(W) = E(W,0)$. Thus, f is nonnegative, and strictly increasing in $0 \leq y < y_c$. Also, $f''(y_c) \neq 0$, and hence the first estimate above holds. For the second one, note that

$$\begin{aligned}
\int |\nabla v|^2 - v^6 &\geq \int |\nabla v|^2 - C_3^6 \left(\int |\nabla v|^2\right)^3 \\
&= \left(\int |\nabla v|^2\right)\left[1 - C_3^6\left(\int |\nabla v|^2\right)^2\right] \\
&\geq \left(\int |\nabla v|^2\right)\left[1 - C_3^6(1-\bar{\delta})^2\left(\int |\nabla W|^2\right)^2\right] \\
&= \int |\nabla v|^2 \left(1 - (1-\bar{\delta})^2\right),
\end{aligned}$$

and the second estimate follows.

Remark: The proof above shows that, if $\|\nabla v\|^2 \leq 3^{1/2}\|\nabla W\|^2$, then $\mathcal{E}(v) = E(v,0) \geq 0$.

We now turn to some properties of general solutions of (EE) and their Lorentz transforms, which are traveling wave solutions of (NLW). It is well known that if $Q \in \Sigma$, then $Q \in C^\infty(\mathbb{R}^3)$. Moreover, $\forall\, \alpha \in \mathbb{N}^3$, $\exists\, C_\alpha > 0$ such that

$$|\partial_x^\alpha Q(x)| \leq C_\alpha\, |x|^{-1-|\alpha|}, \quad |x| \geq 1.$$

From this, one can show that if $Q \in \Sigma$, $1 \leq j \leq 3$, we have

$$\|\partial_{x_j} Q\|^2 = \frac{1}{3}\|\nabla Q\|^2,$$

as can be seen by multiplying the elliptic equation by $x_j\, \partial_{x_j} Q$ and integrating by parts using the above decay estimates.

If $Q \in \Sigma$, $|\vec{\ell}| < 1$, we define

$$\begin{aligned} Q_{\vec{\ell}}(x,t) &= Q\left(\left(\frac{-t}{\sqrt{1-|\vec{\ell}|^2}} + \frac{1}{|\vec{\ell}|^2}\left(\frac{1}{\sqrt{1-|\vec{\ell}|^2}} - 1\right)\vec{\ell}\cdot x\right)\vec{\ell} + x\right) \\ &= Q_{\vec{\ell}}\,(x - t\vec{\ell}, 0), \end{aligned}$$

which shows that $Q_{\vec{\ell}}$ is a global in time, non-scattering solution of (NLW), traveling in the direction $\vec{\ell}$. It is easy to see that, after rotation in the x variable, $Q_{\vec{\ell}} = \tilde{Q}_{\ell\vec{e}_1} = \tilde{Q}_{\ell}$, where $\ell = \pm|\vec{\ell}|$ and $\tilde{Q} \in \Sigma$, where $\tilde{Q}_{\ell}$ are the solutions introduced earlier. By direct computation, one can show that

$$\|\nabla Q_{\vec{\ell}}(0)\|^2 = \frac{3-2|\vec{\ell}|^2}{3\sqrt{1-|\vec{\ell}|^2}}\|\nabla Q\|^2$$

$$\|\partial_t Q_{\vec{\ell}}(0)\|^2 = \frac{|\vec{\ell}|^2}{3\sqrt{1-|\vec{\ell}|^2}}\|\nabla Q\|^2$$

$$E\left(Q_{\vec{\ell}}(0), \partial_t Q_{\vec{\ell}}(0)\right) = \frac{1}{\sqrt{1-|\vec{\ell}|^2}}\mathcal{E}(Q) = \frac{1}{\sqrt{1-|\vec{\ell}|^2}}E(Q,0).$$

We conclude this discussion with a result that shows that all traveling-wave solutions of (NLW) are of the form $Q_{\vec{\ell}}, |\vec{\ell}| < 1$, $Q \in \Sigma$. This is due to DKM. Thus let $F(x,t) = f(x - \vec{\ell}t)$, with $f \in \dot{H}^1(\mathbb{R}^3)$, and assume that F solves (NLW). After rotation, we can again assume that $F(x,t) = f(x_1 - \ell t, x_2, x_3)$, where $\ell \in \mathbb{R}$. The fact that F solves (NLW) is then equivalent to

$$(1-\ell^2)\partial_{x_1}^2 f + \partial_{x_2}^2 f + \partial_{x_3}^2 f + f^5 = 0.$$

Proposition. (DKM) Let $f \in \dot{H}^1(\mathbb{R}^3)\backslash\{0\}$, $\ell \in \mathbb{R}$. Assume that the equation above holds. Then, $\ell^2 < 1$ and there exists $Q \in \Sigma$, such that $f(x) = Q_{\ell \vec{e}_1}(x,0)$.

Sketch of proof. Assume $\ell^2 > 1$. Without loss of generality, we can assume that $\ell > 1$. Consider the function $u(x,t) = f(x_1 + \ell t, x_2, x_3)$ which is a global in time solution of (NLW). Note that $\nabla u(x,0) = f(x)$, $\partial_t u(x,0) = \ell\, \partial x_1 f$. Find M so large that

$$\int_{|x|\geq M} \left(|\nabla u(x,0)|^2 + |\partial_t u(x,0)|^2 + \frac{|u(x,0)|^2}{|x|^2} \right) dx \leq \varepsilon.$$

By finite speed of propagation, we obtain

$$\int_{|x|\geq \frac{3}{2}M+|t|} \left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \right) dx \leq C\varepsilon.$$

Let $K \subset\subset \{(x_2, x_3)\}$, $a < b$. If $t > 0$ is large, then $x_1 \in (a - \ell t, b - \ell t)$ and $(x_2, x_3) \in K \Rightarrow |x| \geq \ell t - A$, where A is a fixed constant depending on K, a, b. Pick t so large that $\ell t \geq \frac{3}{2} M + t + A$, which is possible since $\ell > 1$. Then,

$$\int_K \int_{a-\ell t}^{b-\ell t} |\nabla u(x,t)|^2 dx \leq C\varepsilon,$$

while $\nabla u(x,t) = \nabla f(x_1 + \ell t, x_2, x_3)$, so the integral equals

$$\int_K \int_a^b |\nabla f(x)|^2 dx,$$

which shows, since $\varepsilon > 0$ is arbitrary, that $f \equiv 0$, a contradiction. Assume $\ell^2 = 1$. In this case, f solves the equation $\Delta_{x'} f + f^5 = 0$, $x' = (x_2, x_3)$, and for almost every x_1, $f(x_1, x') \in \dot{H}^1(\mathbb{R}^2) \cap L^6(\mathbb{R}^2)$. Fix such an x_1 and let $F(x_2, x_3) = f(x_1, x_2, x_3)$. We then shows that $F = 0$, using the Pohozaev identity in dimension 2. By elliptic regularity $F \in C^2(\mathbb{R}^2)$.

Also,

$$\mathrm{div}_{x'}(x'|\nabla F|^2) - 2\,\mathrm{div}((x'\cdot\nabla F)\nabla F) = 2\,x'\cdot\nabla(|F|^6/6).$$

If we integrate this identity over $\mathbb{R}^2$, we obtain

$$\frac{4}{6}\int |F|^6 = 0,$$

which gives the claim. To justify the integration we introduce a cut-off $\varphi(x/R)$ and let $R \to \infty$.

Finally, assume that $\ell^2 < 1$. Let $g(x) = f\left(\sqrt{1-\ell^2}\,x_1, x_2, x_3\right)$. Then, $\Delta g + g^5 = 0$, by the equation verified by f, and the conclusion follows.

The ground-state conjecture and the "concentration-compactness/rigidity theorem method"

Part 1: Concentration-compactness.

We will start out by a precise statement of the ground-state conjecture, proved by KM.

Theorem: Let $(u_0, u_1) \in \dot{H}^1 \times L^2$. Assume that $E(u_0, u_1) < E(W, 0)$. Let u be the corresponding solution of the Cauchy problem, with maximal interval $I = I_{\max}(u) = (T_-(u_0, u_1), T_+(u_0, u_1))$. Then,

- **i)** If $\|\nabla u_0\|^2 < \|\nabla W\|^2$, then $I = \mathbb{R}$ and $\|u\|_{S(\mathbb{R})} < \infty$, so that u scatters in both time directions.
- **ii)** If $\|\nabla u_0\|^2 > \|\nabla W\|^2$, then $-\infty < T_- < T_+ < +\infty$.
- **iii)** There is no such (u_0, u_1) with $\|\nabla u_0\|^2 = \|\nabla W\|^2$.

Sketch of proof. We will start out with iii) which follows from variational arguments.

Claim. If $\|\nabla v\|^2 \leq \|\nabla W\|^2$ and $E(v,0) \leq E(W,0)$, then $\|\nabla v\|^2 \leq \frac{\|\nabla W\|^2}{E(W,0)} E(v,0) = 3E(v,0)$.

From the claim, applying it to u_0, we obtain $E(W,0) \leq E(u_0,0) \leq E(u_0,u_1)$, contradicting our assumption. To prove the claim let $f(y) = \frac{1}{2}y - \frac{1}{6}C_3^6 y^3$, so that $f(0) = 0$, $f(\|\nabla W\|^2) = E(W,0)$ and $f(\|\nabla v\|^2) \leq E(v,0)$. Note that f is concave in $\mathbb{R}^+$. Hence, for $s \in (0,1)$, $f(s\|\nabla W\|^2) \geq sf(\|\nabla W\|^2) = sE(W,0)$. Choose $s = \|\nabla v\|^2/\|\nabla W\|^2$, from which the claim follows.

To establish ii), we will do it under the assumption that $u_0 \in L^2$. This extra assumption can be easily removed from finite speed of propagation. Note first the following:

Claim. If $E(u_0,u_1) \leq (1-\delta_0)\,E(W,0)$, $\|\nabla u_0\|^2 > \|\nabla W\|^2$. Then $\exists\,\bar{\delta}$ such that $\|\nabla u_0\|^2 \geq (1+\bar{\delta})\|\nabla W\|^2$, $\bar{\delta} = \bar{\delta}(\delta_0)$. This is proved in the same way as our variational estimate, working this time for $y > y_c$. The conservation of energy, and the continuity of the flow show that, for $t \in I_{\max}(u)$, $\|\nabla u(t)\|^2 \geq (1+\bar{\delta})\|\nabla W\|^2$. We now use an argument originating in the work of H. Levine. Let us show $T_+ < \infty$. Since $E(u_0,u_1) < E(W,0)$, there exists $\bar{\bar{\delta}} > 0$ such that, for $t \in I$, $E(W,0) \geq E(u(t),\partial_t u(t)) + \bar{\bar{\delta}}$. Hence,

$$\frac{1}{6}\int u(t)^6 \geq \frac{1}{2}\int (\partial_t u(t))^2 + \frac{1}{2}\int |\nabla u(t)|^2 - E(W,0) + \bar{\bar{\delta}},$$

and so

$$\int u(6)^6 \geq 3\int (\partial_t u(t))^2 + 3\int |\nabla u(t)|^2 - 6E(W,0) + 6\bar{\bar{\delta}}.$$

Let $y(t) = \int u^2(t)$, $y'(t) = 2\int u(t)\,\partial_t u(t)$, and a simple calculation using the equation and integration by parts gives $y''(t) = 2\int \left[\partial_t u(t)^2 + u(t)^6 - |\nabla u(t)|^2\right]$. Thus,

$$\begin{aligned} y''(t) &\geq 2\int \partial_t u(t)^2 + 6\int \partial_t u(t)^2 + 4\int |\nabla u(t)|^2 - 12\,E(W,0) + \tilde{\delta} \\ &= 8\int \partial_t u(t)^2 + 4\int |\nabla u(t)|^2 - 4\int |\nabla W|^2 + \tilde{\delta} \\ &\geq 8\int \partial_t u(t)^2 + \tilde{\delta}. \end{aligned}$$

Since we are assuming, by contradiction, that $I \cap [0,\infty) = [0,\infty)$, $\exists\, t_0 > 0$ such that $y'(t) \geq y'(t_0) > 0$, for $t > t_0$. Hence, for $t > t_0$, $y(t)\,y''(t) \geq 8\left(\int \partial_t u(t)^2\right)\left(\int u^2(t)\right) \geq 2\,y'(t)^2$. Hence, $\frac{y''(t)}{y'(t)} \geq 2\,\frac{y'(t)}{y(t)}$, or $y'(t) \geq C_0\,y(t)^2$, for $t > t_0$, which leads to finite time blow-up for $y(t)$, a contradiction.

The proof of i) is what the heart of the matter is. First, note from our variational estimates, that if $(u_0, u_1) \in \dot{H}^1 \times L^2$, $E(u_0, u_1) < (1-\delta_0) E(W, 0)$, $\|\nabla u_0\|^2 < \|\nabla W\|^2$, for all $t \in I$, $\|\nabla u(t)\|^2 < (1-\bar{\delta})\|\nabla W\|^2$, $\int |\nabla u(t)|^2 - u^6(t) \geq \bar{\delta} \int |\nabla u(t)|^2$, $E(u(t), 0) \geq 0$ and

$$E(u(t), \partial_t u(t)) \simeq \|(u(t), \partial_t u(t))\|^2_{\dot{H}^1 \times L^2} \simeq \|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2},$$

where we use also the continuity of the flow and the conservation of energy. We call this "energy trapping". To carry out the concentration-compactness procedure, one considers the statement

$$\text{(SC)} \qquad \text{For all } (u_0, u_1) \in \dot{H}^1 \times L^2, \text{ with } \int |\nabla u_0|^2 < \int |\nabla W|^2,$$

$E(u_0, u_1) < E(W, 0)$, the corresponding solution u has $I = \mathbb{R}$ and $\|u\|_{S(\mathbb{R})} < \infty$.

For a fixed (u_0, u_1) verifying the hypothesis of (i), we say that $SC(u_0, u_1)$ holds if $\|u\|_{S(I)} < \infty$. By the local theory of the Cauchy problem, $\exists\,\bar{\delta} > 0$ such that, if $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq \bar{\delta}$, then $SC(u_0, u_1)$ holds. Moreover, by "energy trapping", there exists $\eta_0 > 0$ such that, if (u_0, u_1) is as in (i) and $E(u_0, u_1) \leq \eta_0$, then $SC(u_0, u_1)$ holds. Also, by "energy trapping", for any (u_0, u_1) as in (i), $E(u_0, u_1) \geq 0$. Thus, there exists $\eta_0 \leq E_C \leq E(W, 0)$ such that, if (u_0, u_1) is as in (i) and $E(u_0, u_1) < E_C$, then $SC(u_0, u_1)$ holds, and E_C is optimal with this property. Thus, (i) is the statement $E_C = E(W, 0)$. If we assume then $E_C < E(W, 0)$, we need to reach a contradiction. The concentration-compactness consists in establishing the following two propositions.

Proposition A. There exists $(u_{0,c}, u_{1,c})$ in $\dot{H}^1 \times L^2$ with $E(u_{0,c}, u_{1,c}) = E_C$ $(< E(W,0))$, $\|\nabla u_{0,c}\|^2 < \|\nabla W\|^2$ and such that the corresponding solution u_c has $\|u_c\|_{S(I)} = +\infty$.

Proposition B. Let u_c be as in Proposition A, $0 \in I$, and assume that $\|u_c\|_{S(I_+)} = \infty$, $I_+ = I \cap [0, \infty)$. Then there exist $x(t) \in \mathbb{R}^3$, $\lambda(t) \in \mathbb{R}^+$, $t \in I_+$ such that $K = \{\vec{v}(x,t) : t \in I_+\}$ verifies that $\bar{K} \subset\subset \dot{H}^1 \times L^2$, where

$$\vec{v}(x,t) = \left(\frac{1}{\lambda(t)^{1/2}} u_c \left(\frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{3/2}} \partial_t u \left(\frac{x - x(t)}{\lambda(t)}, t \right) \right).$$

The contradiction will be reached by showing that the only u with the hypothesis of i) and the conclusion of Proposition B is $u \equiv 0$.

Note that, since $E_C < E(W,0)$, $E_C = (1-\bar{\delta})\,E(W,0)$. By the energy trapping estimates, there exists $\delta_1 = \delta_1(\delta_0) > 0$ such that, if $E(u_0,u_1) \leq (1-\delta_0)\,E(W,0)$ and $\|\nabla u_0\|^2 < \|\nabla W\|^2$, then $\|\nabla u_0\|^2 \leq (1-\delta_1)\|\nabla W\|^2$ and $E(u_0,u_1) \geq \delta_1\left(\|\nabla u_0\|^2 + \|u_1\|^2\right)$. The next step is

Lemma. Assume that $\{(u_{0n},u_{1n})\}_n \in \dot{H}^1 \times L^2$ has

$$E(u_{0n},u_{1n}) \leq (1-\delta_0)\,E(W,0), \|\nabla u_{0n}\|^2 \leq (1-\delta_1)\|\nabla W\|^2.$$

Let (after extraction) $\left(U_L^j, \{\lambda_{jn}, x_{jn}, t_{jn}\}\right)$ be a profile decomposition of $\{(u_{0n},u_{1n})\}_n$. Then $\exists\,\delta_2 > 0$, $\delta_3 > 0$, $c_0 > 0$, $\bar{J}$ large and for $J \geq \bar{J}$, $\bar{n} = \bar{n}(J)$ such that for all $1 \leq j \leq J$, $n \geq \bar{n}$

$$E\left(U_L^j(-t_{jn}/\lambda_{jn}), \partial_t U_L^j(-t_{jn}/\lambda_{jn})\right) \geq c_0 \|(U_0^j, U_1^j)\|^2_{\dot{H}^1 \times L^2}$$

$$E(w_{0n}^J, w_{1n}^J) \geq c_0 \|(w_{0n}^J, w_{1n}^J)\|^2_{\dot{H}^1 \times L^2}$$

$$E\left(U_L^j(-t_{jn}/\lambda_{jn}), \partial_t U_L^j(-t_{jn}/\lambda_{jn})\right) \leq (1-\delta_2)\, E(W,0)$$

$$E(w_{0n}^J, w_{1n}^J) \leq (1-\delta_2)\|\nabla W\|^2$$

$$\|\nabla U_L^j(-t_{jn}/\lambda_{jn})\|^2 \leq (1-\delta_3)\|\nabla W\|^2$$

$$\|\nabla w_{0n}^J\|^2 \leq (1-\delta_3)\|\nabla W\|^2$$

For the proof, we apply the modified version of the Pythagorean expansion, that holds separately for ∂_t and ∇, with $\varepsilon_0 = \frac{\delta_1}{2}\|\nabla W\|^2$. Then, for $J \geq \bar{J}$ and n large, we have $\|\nabla w_{0n}^J\|^2 \leq \left(1 - \frac{\delta_1}{2}\right)\|\nabla W\|^2$, and for j core, $\|\nabla U_L^j(0)\|^2 \leq \left(1 - \frac{\delta_1}{2}\right)\|\nabla W\|^2$, and hence, $E(w_{0n}^J, w_{1n}^J) \geq 0$, and for j core $E(U_L^j(0), \partial_t U_L^j(0)) \geq 0$. For j scattering ($|t_{jn}|/\lambda_{jn} \to \infty$), by well-known linear estimates, $\|U_L^j(-t_{jn}/\lambda_{jn})\|_{L^6} \underset{n}{\longrightarrow} 0$. Hence, for $J \geq \bar{J}$, $n = n(J)$ large, $1 \leq j \leq J$, j scattering, we have

$$E\left(U_L^j(-t_{jn}/\lambda_{jn}), \partial_t U_L^j(-t_{jn}/\lambda_{jn})\right) \geq \frac{2}{5}\|\nabla U_L^j(-t_{jn}/\lambda_{jn})\|^2.$$

Applying now the standard Pythagorean expansion of the non-linear energy,

$$E(u_{0n}, u_{1n}) = \sum_{j=1}^{J} E\left(\vec{U}_L^j(-t_{jn}/\lambda_{jn})\right) + E(w_{0n}^J, w_{1n}^J) + o_n(1).$$

Using the non-negativity of all the energies, we see that

$$E(U_L^j(0), \partial_t U_L^j(0)) \leq (1-\delta_0/2)\, E(W,0), \quad \text{for } j \text{ core},$$

$$E\left(U_L^j(-t_{jn}/\lambda_{jn}, \partial_t U_L^j(-t_{jn}/\lambda_{jn})\right) \leq (1-\delta_0/2)\, E(W,0), \text{for } j \text{ scattering},$$

and

$$E(w_{0n}^J, w_{1n}^J) \leq (1-\delta_0/2)\, E(W,0).$$

Also, for j scattering,

$$\frac{2}{5}\|\nabla U_L^j(-t_{jn}/\lambda_{jn})\|^2 \leq (1-\delta_0/2)\, E(W,0) \leq \frac{1}{3}\|\nabla W\|^2,$$

and so, for suitable δ_3, for n large, we have, for all $1 \leq j \leq J$

$$\|\nabla U_L^j(-t_{jn}/\lambda_{jn})\|^2 \leq (1-\delta_3)\|\nabla W\|^2, \|\nabla w_{0n}^J\|^2 \leq (1-\delta_3)\|\nabla W\|^2.$$

This gives all the desired upper bounds, while the lower bounds follow by energy trapping.

We next use:

Lemma. Let $\{(u_{0n}, u_{1n})\}_n$ be a sequence in $\dot{H}^1 \times L^2$ such that $E(u_{0n}, u_{1n}) \to E_C$, $\|\nabla u_{0n}\|^2 < \|\nabla W\|^2$, with maximal interval I_n, $0 \in I_n$, and such that

$$\|u_n\|_{S(I_{n_+})} = \infty, \lim_{n\to\infty} \|u_n\|_{S(I_{n_-})} = \infty.$$

Let, after extraction, $(U_L^j, \{\lambda_{jn}, x_{jn}, t_{jn}\}_n)_j$ be a profile decomposition of $\{(u_{0n}, u_{1n})\}_n$. Then, (after reordering the profiles) $U_L^j \equiv 0$ for all $j \geq 2$, there exists C_0 such that $|t_{1n}/\lambda_{1n}| \leq C_0$ and $\|(w_{0n}^J, w_{1n}^J)\|_{\dot{H}^1\times L^2} \to 0$.

Proof. By our variational estimates, since $E_C < E(W,0)$, we can fix δ_0, δ_1 so that $E(u_{0n}, u_{1n}) \leq (1-\delta_0)E(W_0)$ and $\|\nabla u_{0n}\|^2 \leq (1-\delta_1)\|\nabla W\|^2$ for large n. By the Lemma, assuming that both U_L^1 and U_L^2 are non-zero and hence, for some $\varepsilon > 0$ $\|(U_0^j, U_1^j)\|^2_{\dot{H}^1\times L^2} \geq \varepsilon$ $j = 1,2$ and thus $\|U_L^j(-t_{jn}/\lambda_{jn}), \partial_t U_L^j(-t_{jn}/\lambda_{jn})\|^2_{\dot{H}^1\times L^2} \geq \varepsilon$, we have $E(U_L^j(-t_{jn}/\lambda_{jn}), \partial_t U_L^j(-t_j/\lambda_{jn})\|_{\dot{H}^1\times L^2} \geq \tilde{\varepsilon}, j = 1,2$. By the Pythagorean expansion of E and the non-negativity of all the energies, for any j, and n large, we now have

$$E(U_L^j(-t_{jn}/\lambda_{jn}), \partial_t U_L^j(-t_{jn}/\lambda_{jn})) \leq E_C - \tilde{\varepsilon}/2$$

and

$$\|\nabla U_L^j(-t_{jn}/\lambda_{jn})\|^2 < \|\nabla W\|^2.$$

By the optimality of E_C, all the nonlinear profiles U^j scatter. By the approximation theorem, this contradicts $\|u_n\|_{S(I_{n+})} = +\infty$. The same argument shows that $\|(w_{0n}^J, w_{1n}^J)\|_{\dot{H}^1\times L^2} \to 0$. Next, since $\|u_n\|_{S(I_{n+})} = \infty$, we must have $-t_{1n}/\lambda_{1n} \le C_0$. Assume that $-t_{1n}/\lambda_{1n} \to -\infty$. Then, for large n, $\left\| \frac{1}{\lambda_{1n}^{1/2}} U_L^1 \left(\frac{x-x_{1n}}{\lambda_{1n}}, \frac{t-t_{1n}}{\lambda_{1n}} \right) \right\|_{S(-\infty,0)} \le \frac{\bar{\delta}}{2}$, where $\bar{\delta}$ is as in the local theory of the Cauchy problem. Hence, by this local theory, $\|u_n\|_{S(I_{n-})} \le C\bar{\delta}$, which contradicts $\lim_n \|u_n\|_{S(I_{n-})} = \infty$.

Proof of Proposition A. By definition of E_C we can choose (possible using time translation) a sequence $\{(u_{0n}, u_{1n})\}$ as in the last Lemma. Let U_L^1 be the non-zero profile, U^1 the non-linear profile. If $\|U^1\|_{S(I_1^+)} < \infty$, the approximation theorem gives that $\|u_n\|_{S(I_{n+})} < \infty$, a contradiction. Moreover, $E(U_L^1(-t_{1n}/\lambda_{1n}), \partial_t U_L^1(-t_{1n}/\lambda_{1n})) \to E_C$ by the Pythagorean expansion, since $\|(w_{0n}^J, w_{1n}^J)\|_{\dot{H}^1 \times L^2} \to 0$. But then, by the invariance of E, $E(U^1, \partial_t U^1) = E_C$. Since $\|\nabla U_L^1(-t_{1n}/\lambda_{1n})\|^2 \leq (1-\delta_3)\|\nabla W\|^2$ for n large, for n large $\|\nabla U^1(-t_{1n}/\lambda_{1n})\|^2 < \|\nabla W\|^2$. By the variational estimates, if $\bar{t} \in I_1$, $\|\nabla U^1(\bar{t})\|^2 < (1-\tilde{\delta}_3)\|\nabla W\|^2$, and we let $u_c = U^1$.

Proof of Proposition B. By continuity of the flow, it suffices to show that if $t_n \to T_+(u_C)$, after extraction, we can find λ_n, x_n such that $\left(\frac{1}{\lambda_n^{1/2}} u_c\left(\frac{x-x_n}{\lambda_n}, t_n\right), \frac{1}{\lambda_n^{3/2}} \partial_t u_c\left(\frac{x-x_n}{\lambda_n}, t_n\right)\right)$ converges. Consider $\{(u_c(t_n), \partial_t u_c(t_n)\}$. This is a sequence verifying the assumptions of the Lemma. By the conclusion of the second Lemma, after extraction and time translating U_L^1, we can assume $t_{1n} \equiv 0$. Thus,

$$
\begin{aligned}
(u_c(t_n), \partial_t u_c(t_n)) &= \left(\frac{1}{\lambda_{1n}^{1/2}} U_L^1\left(\frac{x-x_{1n}}{\lambda_{1n}}, 0\right), \frac{1}{\lambda_{1n}^{3/2}} \partial_t U_L^1\left(\frac{x-x_{1n}}{\lambda_{1n}}, 0\right)\right) \\
&+ (w_{0n}^J, w_{1n}^J),
\end{aligned}
$$

with $\|(w_{0n}^J, w_{1n}^J)\|_{\dot{H}^1\times L^2} \underset{n}{\longrightarrow} 0$, which gives the result.

Part 2: Rigidity Theorem.

We start by giving some further properties of critical elements because of the continuity of $(u_C(t), \partial_t u_C(t))$ in $\dot{H}^1 \times L^2$, we can always construct $x(t), \lambda(t)$ as in Proposition B, with $x(t), \lambda(t)$ continuous on I_+.

Lemma: Let u_C be a critical element as in Proposition B. Then there is a (possibly different) critical element v_C, with corresponding $\tilde{\lambda}(t), \tilde{x}(t)$ and $A_0 > 0$ so that $\tilde{\lambda}(t) \geq A_0$ for $t \in [0, T_+(v_C))$.

Sketch of proof: If $\lambda(t)$ does not have the required property, $\exists \{t_n\} \subset [0, T_+(u_C))$ such that $\lambda(t_n) \to 0$. It is not difficult to see that, after extraction $\{t_n\} \to t_0 = T_+(u_C)$. (If not $t_0 \in [0, T_+(u_C))$, $\lambda(t_n) \to 0$. But then, by the compactness property of K $(u_C(x,t_n), \partial_t u_C(x,t_n)) - (\lambda(t_n)^{1/2} w_0(\lambda(t_n)x + x(t_n)), \lambda(t_n)^{3/2} w_1(\lambda(t_n)x + x(t_n))) \to 0$, where $(w_0, w_1) \in \bar{K}$. But, since $\lambda(t_n) \to 0$, and $(u_C(x,t_n), \partial_t u_C(x,t_n)) \to (u_C(x,t_0), \partial_t u_C(x,t_0))$, we must have $(w_0, w_1) = (0,0)$. But then $(0,0) \in \bar{K}$, which forces u_C to scatter, a contradiction.) After possibly redefining $\{t_n\}$, we can assume that $\frac{1}{2}\lambda(t_n) \leq \inf_{[0,t_n]} \lambda(t)$. By compactness of $\bar{K}$,

$$(w_{0n}(x), w_{1n}(x)) = \left(\frac{1}{\lambda(t_n)^{1/2}} u_C\left(\frac{x - x(t_n)}{\lambda(t_n)}, t_n\right), \right.$$

$$\left. \frac{1}{\lambda(t_n)^{3/2}} \partial_t u_C\left(\frac{x - x(t_n)}{\lambda(t_n)}, t_n\right) \right) \xrightarrow[n\to\infty]{} (w_0, w_1) \in \dot{H}^1 \times L^2.$$

One can then see that (w_0, w_1) is the desired critical element, with $\tilde{\lambda}(\tau)$ defined for $\tau \in (T_-(w), 0]$ as

$$\tilde{\lambda}(\tau) = \lim_{n} \frac{\lambda(t_n + \tau(\lambda(t_n)))}{\lambda(t_n)} \text{ and } \tilde{x}(\tau) = \lim_{n\to\infty} \frac{x(t_n) + \tau(\lambda(t_n)) - x(t_n)}{\lambda(t_n)}.$$

Note that $\tilde{\lambda}_n(\tau) \geq 1/2$. All the limits can be shown to exist after extraction.

Lemma: Let u_C be as in Proposition A, $T_+ < \infty$ (and after scaling $T_+ = 1$). Then $\exists\, C_0 = C_0(\bar{K})$ such that

$$\lambda(t) \geq C_0(\bar{K})/1 - t.$$

Proof: Consider $t_n \uparrow 1$,

$$\vec{u}_n(0) = \left(\frac{1}{\lambda(t_n)^{1/2}} u_C \left(\frac{x - x(t_n)}{\lambda(t_n)}, t_n \right), \frac{1}{\lambda(t_n)^{3/2}} \partial_t u_C \left(\frac{x - x(t_n)}{\lambda(t_n)}, t_n \right) \right).$$

By a corollary of the Perturbation Theorem, since $\bar{K}$ is compact, $\exists\, C_0 = C_0(\bar{K})$ such that $T_+(u_n) \geq C_0$. Since

$$(u_C(y_1, t_n), \partial_t u_C(y_1, t_n)) =$$
$$(\lambda(t_n)^{1/2} u_{0n}(\lambda(t_n) y + x(t_n)), \lambda(t_n)^{3/2} u_{1n}(\lambda(t_n) y + x(t_n))),$$

for $0 < t_n + t < 1$ we have $u(y, t_n + t) = \lambda(t_n)^{1/2}\, u_n(\lambda(t_n) y + x(t_n), \lambda(t_n) t)$. Thus, we have $t_n + t < 1$ whenever $0 < \lambda(t_n) t \leq C_0$. Choose $t = C_0 / \lambda(t_n)$ and thus $\lambda(t_n) \geq C_0 / (1 - t_n)$ as desired.

Lemma: Let u_C be as in previous Lemma. Then, there exists $\bar{x} \in \mathbb{R}^3$ such that $\operatorname{supp} \vec{u}_C(t) \subset B(\bar{x}, 1-t)$ $0 < t < 1$.

Sketch of proof: By finite speed, if $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq A$, $\varepsilon > 0$, $M > 0$ are such that $\int_{|x| \geq M} |\nabla u_0|^2 + u_1^2 + \frac{|u_0|^2}{|x|^2} \leq \varepsilon$, then, for $0 < t < T_+$

$$\int_{|x| \geq \frac{3}{2}M + t} |\nabla u(t)|^2 + |\partial_t u(t)|^2 + \frac{|u(t)|^2}{|x|^2} + |u(t)|^6 \leq C\varepsilon.$$

For the proof, we show that for any $R_0 > 0$

$$\lim_{t \uparrow 1} \int_{|x + x(t)/\lambda(t)| > R_0} |\nabla u_C(x,t)|^2 + |\partial_t u_C(x,t)|^2 + \frac{|u_C(x,t)|^2}{|x|^2} = 0.$$

This is an easy consequence of $\bar{K}$ compact, $\lambda(t) \to \infty$ as $t \to 1$.

Using the first remark backward in time, we then see that for each $s \in [0,1)$,

$$\lim_{t\uparrow 1}\int_{\left|x+\frac{x(t)}{\lambda(t)}\right|\geq \frac{3}{2}R_0+(t-s)} |\nabla u_C(x,s)|^2+|\partial_t u_C(x,s)|^2=0.$$

This implies $\left|\frac{x(t)}{\lambda(t)}\right| \leq M$, $0<t<1$. If not, $\exists\, t_n \uparrow 1$ such that $\left|\frac{x(t_n)}{\lambda(t_n)}\right| \to \infty$. But then, for $R>0$, $\{|x|\leq R\} \subset \left\{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right| \geq \frac{3}{2}R+t_n\right\}$ for n large, so that, taking $s=0$ and passing to the limit in n, $\int_{|x|\leq R} |\nabla u_{0C}|^2+|u_{1C}|^2=0$, a contradiction. Pick now $t_n \to 1$, $\frac{x(t_n)}{\lambda(t_n)} \to -\bar{x}$. For every $\eta_0>0$, for n large, $s\in[0,1)$

$$\{|x-\bar{x}|\geq 1+\eta_0-s\} \subset \left\{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right| \geq \frac{3}{2}R_0+(t_n-s)\right\}$$

for $R_0=R_0(\eta_0)>0$.

But then,

$$\int_{|x-\bar{x}|\geq 1+\eta_0-s} |\nabla u_C(x,s)|^2 + |\partial_t u_C(x,s)|^2 = 0,$$

which gives the claim.

In this situation we can assume, without loss of generality, that $\bar{x} = 0$.

There is another conservation law: $\forall\, t \in I$, $\int \nabla u(t)\, \partial_t u(t) = \int \nabla u_0 \cdot u_1$. Conservation of momentum.

We turn to a fundamental property of critical elements, in order to treat the non-radial case.

Theorem: Let $(u_{0,C}, u_{1,C})$ be as in Proposition A. Assume that $x(t)$, $\lambda(t)$ are continuous. Assume that either $T_+ < \infty$ or $T_+ = \infty$, $\lambda(t) \geq A > 0$. Then $\int \nabla u_{0,C}, u_{1,C} = 0$.

Sketch of proof: We exploit Lorentz invariance. Assume, without loss of generality, that $\int \partial x_1\, u_{0,C} \cdot u_{1,C} = \gamma > 0$. Consider first $T_+ < \infty$, and hence without loss, $T_+ = 1$. Recall that $\operatorname{supp} \vec{u}_C(t) \subset B(0, 1-t)$, $0 < t < 1$. For convenience, set $u(x,t) = u_C(x, 1+t)$, $-1 < t < 0$, and $\vec{u}(t)$ supported in $B(0, |t|)$. For $0 < \ell < 1$, small consider the Lorentz transform $z_\ell(x_1, x', t) = u\left(\frac{x_1 - \ell t}{\sqrt{1-\ell^2}}, x', \frac{t - \ell x_1}{\sqrt{1-\ell^2}}\right)$. Using the support property of u, we see that z_ℓ solves (NLW) in $-\frac{1}{2} \leq t < 0$. An easy calculation shows that $\operatorname{supp} \vec{z}_\ell(t) \subset B(0, |t|)$, which immediately show that 0 is the final time of existence of z_ℓ, since it is not identically 0. A lengthy calculation gives:

$$\lim_{\ell \downarrow 0} \frac{E(\vec{z}_\ell(-1/2)) - E(\vec{u}_C)}{\ell} = -\gamma.$$

Moreover, for some $t_0 \in [-1/2, -1/4]$, $\int |\nabla z_\ell(t_0)|^2 < \int |\nabla W|^2$, for ℓ small, by integrating in that region of t, change of variables and the variational estimate. But, since $E(\vec{u}_C) = E_C$, for ℓ small, this and the fact that $T_+(z_\ell) = 0$ contradicts the definition of E_C.

For $T_+ = +\infty$, it is easiest to do a renormalization. We first rescale u_C and consider for R large, ℓ small and positive $z_{\ell,R}(x,t) = U_R\left(\frac{x_1-\ell t}{\sqrt{1-\ell^2}}, x', \frac{t-\ell x_1}{\sqrt{1-\ell^2}}\right)$, $u_R(x,t) = R^{1/2}u_C(Rx,Rt)$.

To derive a contradiction, we let $h(t_0) = \theta(x)\, z_{\ell,R}(x_1,x',t_0)$, where θ is a fixed cut-off function, and show, by integration in $t_0 \in (1,2)$, that for small ℓ and large R, we have, for some $t_0 \in (1,2)$, $E(\vec{h}(t_0)) < E_C - \frac{1}{2}\gamma\ell$ and $\int |\nabla h(t_0)|^2 < \int |\nabla W|^2$. We then let v be the solution of (NLW) with data $\vec{h}(t_0)$. By the properties of E_C and a concentration-compactness argument, we show that $\|v\|_{S(-\infty,+\infty)} \leq C\left(\frac{1}{2}\gamma\ell\right)$, for all R large. But, since $\|u_C\|_{S(0,\infty)} = +\infty$, we obtain that

$$\|u_R\|_{L^8_{[0,1]}L^8(|x|<1)} \xrightarrow[R\to\infty]{} \infty.$$

Now, by finite speed of propagation, $v = z_{\ell,R}$ on a large set, and after a change of variables to undo the Lorentz transform, we reach a contradiction. The details are technical. To reach a contradiction in the proof of i) in the Theorem, we are now reduced to proving the following Rigidity Theorem.

Theorem: Suppose that $E(u_0,u_1) < E(W,0)$, $\int |\nabla u_0|^2 < \int |\nabla W|^2$, and u is the corresponding solution of (NLW), $I_+ = [0,T_+)$. Assume that

a) $\int \nabla u_0 \, u_1 = 0$

b) $\exists \, x(t), \lambda(t), t \in I_+$ such that

$$K = \left\{ \frac{1}{\lambda(t)^{1/2}} \, u\left(\frac{x - x(t)}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{3/2}} \, \partial_t \, u\left(\frac{x - x(t)}{\lambda(t)}, t\right), t \in I_+ \right\}$$

has $\bar{K} \subset\subset \dot{H}^1 \times L^2$.

c) $x(t), \lambda(t)$ are continuous, $\lambda(t) > 0$. If $T_+ < \infty$, we have $\lambda(t) \geq C/(T_+ - t)$, $\operatorname{supp} \vec{u}(t) \subset B(0, T_+ - t)$. If $T_+ = \infty$, we have $\lambda(t) \geq A_0 > 0$, $x(0) = 0$, $\lambda(0) = 1$.

Then $u \equiv 0$.

Sketch of proof: We start with the case $T_+ = \infty$. We need some well-known integral identities. Choose $\Phi \in C_0^\infty(B_1)$, $\Phi \equiv 1$ on B_1, $\Phi_R(x) = \Phi(x/R)$, $\psi_R(x) = x\,\Phi_R(x)$. For u a solution of (NLW), $t \in I$, we let

$$r(R) = r(t,R) = \int_{|x|\geq R} \left\{ |\nabla u|^2 + |\partial_t u|^2 + \frac{|u|^2}{|x|^2} + |u|^6 \right\}.$$

Lemma:

i) $\partial_t \int \psi_R \nabla u\, \partial_t u = -\frac{3}{2} \int (\partial_t u)^2 + \frac{1}{2} \int [|\nabla u|^2 - u^6] + 0(r(R))$

ii) $\partial_t \int \Phi_R\, u\, \partial_t u = \int (\partial_t u)^2 - \int |\nabla u|^2 + \int u^6 + 0(r(R))$

iii) $\partial_t \int \psi_R \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{6}u^6 \right\} = -\int \nabla u\, \partial_t u + 0(r(R)).$

We now start the sketch of proof for $T_+ = +\infty$. Observe that, if $(u_0, u_1) \neq (0,0)$, $E = E(u_0, u_1)$, by the variational estimates $E > 0$, $\sup_{t>0} \|\vec{u}(t)\|^2_{\dot{H}^1 \times L^2} \leq CE$. Also, $\int |\nabla u(t)|^2 - \int u^6(t) \geq C \int |\nabla u(t)|^2$, $t > 0$ and $\frac{1}{2} \int \partial_t u(t)^2 + \frac{1}{2} \int \left[|\nabla u(t)|^2 - u^6(t) \right] \geq CE$, $t > 0$. The compactness of $\bar{K}$, and the fact that $\lambda(t) \geq A_0$, and a change of variables, show that given $\varepsilon > 0$, $\exists R_0(\varepsilon) \geq 0$ such that for all $t > 0$ we have

$$(+) \qquad \int_{\left|x + \frac{x(t)}{\lambda(t)}\right| \geq R_0(\varepsilon)} |\partial_t u|^2 + |\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^6 \leq \varepsilon E.$$

The proof of this case is carried out through:

Lemma 1: $\exists\, \varepsilon_1 > 0$, $C > 0$ such that, if $0 < \varepsilon < \varepsilon_1$, $R > 2R_0(\varepsilon)$, $\exists\, t_0 = t_0(R, \varepsilon)$, $0 < t_0 \le CR$, such that for $0 < t < t_0$

$$\left|\frac{x(t)}{\lambda(t)}\right| < R - R_0(\varepsilon) \quad \text{and} \quad \left|\frac{x(t_0)}{\lambda(t_0)}\right| = R - R_0(\varepsilon).$$

Lemma 2: $\exists\, \varepsilon_2 > 0$, $R_1(\varepsilon) > 0$, $C_0 \ge 0$ such that if $R > R_1(\varepsilon)$, for $0 < \varepsilon < \varepsilon_2$ we have $t_0(R, \varepsilon) \ge C_0\, R/\varepsilon$.

Note that Lemma 1 + Lemma 2 give a contradiction for ε small, R large.

Sketch of proof of Lemma 1: Since $x(0) = 0$, $\lambda(0) = 1$, if not, $\forall\, 0 < t < CR$, C large, we have $\left|\frac{x(t)}{\lambda(t)}\right| < R - R_0(\varepsilon)$. Let $z_R(t) = \int \psi_R \nabla u\, \partial_t u + \int \phi_R\, u\, \partial_t u$. Then, $z_R'(t) = -\frac{1}{2}\int \partial_t u(t)^2 - \frac{1}{2}\int \left[|\nabla u(t)|^2 - u^6(t)\right] + 0(r(R))$. But, for $|x| > R$, $0 < t < CR$, we have $\left|x + \frac{x(t)}{\lambda(t)}\right| \ge R_0(\varepsilon)$, so that $|r(R)| \le \tilde{C}\varepsilon E$. Thus, for ε small, we have $z_r'(t) \le -\tilde{\tilde{C}}E$. By the variational estimates, we have $|z_R(t)| \le C_1 RE$. Integrating in t, we obtain $CR\tilde{\tilde{C}}E \le 2C_1 RE$, which is a contradiction for C large.

Sketch of proof of Lemma 2: For $0 < t < t_0$, set $y_R(t) = \int \psi_R \left\{\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{6}|u|^6\right\}$. For $|x| > R$, $\left|x + \frac{x(t)}{\lambda(t)}\right| \geq R_0(\varepsilon)$, and since $\int \nabla u_0\, u_1 = 0 = \int \nabla u(t)\, \partial_t u(t)$, $y_R'(t) = 0(r(R))$ and thus $|y_R(t_0) - y_R(0)| \leq \tilde{C}\varepsilon\, E\, t_0$. But,

$$|y_R(0)| \leq \tilde{C}R_0(\varepsilon)\, E + 0(Rr(R_0(\varepsilon))) \leq \tilde{C}E[R_0(\varepsilon) + \varepsilon\, R].$$

Also,

$$|y_R(t_0)| \geq \left|\int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right| \leq R_0(\varepsilon)} \psi_R \left\{\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{6}\, u^6\right\}\right| - \left|\int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right| \geq R_0(\varepsilon)} \cdots\right|.$$

In the first integral, $|x| \leq R$, so $\psi_R(x) = x$. The second integral is bounded by $MR\varepsilon E$. Thus,

$$|y_R(t_0)| \geq \left|\int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right| \leq R_0(\varepsilon)} x \left\{\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{6}u^6\right\}\right| - MR\varepsilon E.$$

The integral in the right equals

$$-\frac{x(t_0)}{\lambda(t_0)}\cdot\int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right|\le R_0(\varepsilon)}\{\dots\}+\int_{\left|x+\frac{x(t_0)}{\lambda(t_0)}\right|\le R_0(\varepsilon)}\left(x+\frac{x(t_0)}{\lambda(t_0)}\right)\{\dots\},$$

so its absolute value is greater than or equal to

$$(R-R_0(\varepsilon))E-\tilde{C}(R-R_0(\varepsilon))\varepsilon E-\tilde{C}R_0(\varepsilon)E.$$

Thus,

$$|y_R(t_0)|\ge E(R-R_0(\varepsilon))[1-\tilde{C}\varepsilon]-\tilde{C}R_0(\varepsilon)E-MR\varepsilon E\ge ER/4$$

for ε small, R large.

But then, $\frac{ER}{4}-\tilde{C}E[R_0(\varepsilon)+\varepsilon R]\le\tilde{C}\varepsilon E\,t_0$, which yields the Lemma for R large, ε small.

We now turn to $T_+ = 1$, supp $\vec{u}(t) \subset B(0, 1-t)$, $\lambda(t) \geq C/1-t$, $\int \nabla u_0\, u_1 = 0$. The first step is to reduce to the self-similar case.

Lemma: $\exists\, C > 0$ such that $\lambda(t) \leq C/1-t$.

Proof: If not $\exists\, t_n \uparrow 1$ such that $\lambda(t_n)(1-t_n) \to \infty$. Let $z(t) = \int x \nabla u\, \partial_t u + \int u\, \partial_t u$. This is well defined because of the support property of u. For $0 < t < 1$,

$$z'(t) = -\frac{1}{2} \int \partial_t u(t)^2 - \frac{1}{2} \int \left[|\nabla u(t)|^2 - u^6(t) \right].$$

Again by variational estimates, $E > 0$, $\sup\limits_{0<t<1} \|\vec{u}(t)\|^2_{\dot{H}^1 \times L^2} \leq CE$. Also, $z'(t) \leq -CE$, $0 < t < 1$. From the support property of u, $\lim\limits_{t \to 1} z(t) = 0$. Hence, integrating in t we obtain $z(t) \geq CE(1-t)$, $0 < t < 1$.

We will next show $\frac{z(t_n)}{(1-t_n)} \to 0$, to reach a contradiction. Since

$$\int \nabla u(t)\, \partial_t u(t) = 0, \frac{z(t_n)}{(1-t_n)} = \int \frac{(x + x(t_n)/\lambda(t_n))\nabla u \cdot \partial_t u}{(1-t_n)} + \int \frac{u\, \partial_t u}{(1-t_n)}.$$

Now, for $\varepsilon > 0$ given,

$$\int_{|x+x(t_n)/\lambda(t_n)|\leq \varepsilon(1-t_n)} |x+x(t_n)/\lambda(t_n)|\, |\nabla u(t_n)|\, |\partial_t u(t_n)| + |u(t_n)|\, |\partial_t u(t_n)| \leq C\varepsilon E(1-t_n)$$

(for the second term we apply Hardy). We next show $\left|\frac{x(t_n)}{\lambda(t_n)}\right| \leq 2(1-t_n)$. If not, $B\left(-\frac{x(t_n)}{\lambda(t_n)}, (1-t_n)\right) \cap B(0,(1-t_n)) = \emptyset$, so that $\int_{B(-x(t_n)/\lambda(t_n),(1-t_n))} |\nabla u(t_n)|^2 + |\partial_t u(t_n)|^2 = 0$, while

$$\int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right|\geq(1-t_n)} |\nabla u(t_n)|^2 + |\partial_t u(t_n)|^2 =$$

$$\int_{|y|\geq\lambda(t_n)(1-t_n)} \left|\nabla_{y,t}\, u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)\right|^2 \frac{dy}{\lambda(t_n)^3} \xrightarrow[n\to\infty]{} 0$$

by the compactness of $\bar{K}$, since $\lambda(t_n)(1-t_n) \to \infty$. This contradicts $E > 0$. Then:

$$\frac{1}{(1-t_n)} \int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right|\geq\varepsilon(1-t_n)} \left|x + \frac{x(t_n)}{\lambda(t_n)}\right| |\nabla u(t_n)|\, |\partial_t u(t_n)| \leq$$

$$\leq 3 \int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right|\geq\varepsilon(1-t_n)} |\nabla u(t_n)|\, |\partial_t u(t_n)| \xrightarrow[n]{} 0,$$

since $\lambda(t_n)(1-t_n) \to \infty$. Arguing similarly for $\int_{\left|x+\frac{x(t_n)}{\lambda(t_n)}\right|\geq\varepsilon(1-t_n)} u\, \partial_t u\, dx/1-t_n$ (using the Hardy inequality), the proof follows.

Proposition: Let u be as in the Rigidity Theorem, $T_+ = 1$, $\operatorname{supp} \vec{u}(t) \subset B(1-t)$. Then,

$$K = \{((1-t)^{1/2}\, u((1-t)x,t)), (1-t)^{3/2}\, \partial_t\, u((1-t)x,t), 0 \le t < 1\}$$

has $\bar{K} \subset\subset \dot{H}^1 \times L^2$.

For the proof, note that if $\tilde{K} \subset\subset L^2(\mathbb{R}^3)^4$, then so is $K_1 = \{\lambda^{3/2}\, \vec{v}(\lambda,x) : \vec{v} \in \tilde{K}, c_0 \le \lambda \le C_0\}$. Since $c_0 \le (1-t)\,\lambda(t) < C_0 = \{\vec{v}(x,t) = ((1-t)^{3/2}\, \nabla u((1-t)[x-x(t)],t), (1-t)^{3/2}\, \partial_t\, u((1-t)(x-x(t)),t))\}$ has compact closure on $L^2(\mathbb{R}^3)^4$. Since $\vec{v}(\cdot,t)$ is supported in $|x-x(t)| \le 1$, and $E > 0$, so that $\vec{v}(\cdot,t)$ is never 0, the above precompactness gives $|x(t)| \le M$, and this gives the result.

The final step is to show that there is no solution as in the Proposition, which concludes the proof. This is the fact that no self-similar compact blow-up solution exists. This is a fundamental fact in the study, which proves to be crucial throughout the theory.

To prove this, we introduce self-similar variables, following Giga-Kohn in the parabolic case and Merle-Zaag in the wave case in a subcritical regime. We introduce now the self-similar variables: $y = x/1-t$, $s = \log 1/1-t$, define

$$w(y,s) = w(y,s;0) = (1-t)^{1/2}u(x,t) = e^{-s/2}u(e^{-s}y, 1-e^{-s}).$$

This is defined for $0 \leq s < \infty$, and $\operatorname{supp} \vec{w}(-,s) \subset \{|y| \leq 1\}$. We also consider, for $\delta > 0$, $u_\delta(x,t) = u(x,t+\delta)$ also a solution of (NLW), and its corresponding w, which we denote by $w(y,s;\delta)$. Thus, $y = \frac{x}{1+\delta-t}$, $s = \log \frac{1}{1+\delta-t}$,

$$w(y,s;\delta) = (1+\delta-t)^{1/2}u(x,t) = e^{-s/2}u(e^{-s}y, 1+\delta-e^{-s}).$$

$w(y,s;\delta)$ is defined for $0 \leq s < -\log\delta$ and

$$\operatorname{supp} \vec{w}(-,s;\delta) \subset \left\{ |y| \leq \frac{e^{-s}-\delta}{e^{-s}} = \frac{1-t}{1+\delta-t} \leq 1-\delta \right\}.$$

The w solve, where they are defined, the equation

$$\partial_s^2 w = \frac{1}{\rho}\mathrm{div}(\rho\nabla w - \rho(y\cdot\nabla w)y) - \frac{3}{4}w + w^5 - 2y\cdot\nabla\partial_s w - 2\partial_s w, \rho(y) = (1-|y|^2)^{-1/2}.$$

The static part of this equation is degenerate elliptic, degenerating at $|y| = 1$. In fact,

$$\frac{1}{\rho}\mathrm{div}(\rho\nabla w - \rho(y\cdot\nabla w)y) = \frac{1}{\rho}\mathrm{div}(\rho(I - y\otimes y)\nabla w),$$

which is elliptic with smooth coefficients for $|y| < 1$, but degenerates at $|y| = 1$.

The following are straightforward bounds for $w(\cdot;\delta)$, $\delta > 0$:

$$w \in H_0^1(B_1), \int_{B_1} |\nabla w|^2 + |\partial_s w|^2 + |w|^6 \le C.$$

Moreover, by Hardy's inequality for $H_0^1(B_1)$ functions, we have:

$$\int_{B_1} \frac{w^2}{(1-|y|^2)^2}\,dy \le C.$$

These bounds are uniform in $\delta > 0$, $0 < s < -\log\delta$. Next, following Merle-Zaag we have a monotonic energy:

$$\begin{aligned}\tilde{E}(w(s;\delta)) &= \int_{B_1} \frac{1}{2}\left\{(\partial_s w)^2 + |\nabla w|^2 - (y\cdot\nabla w)^2\right\}\frac{dy}{(1-|y|^2)^{1/2}} \\ &+ \int_{B_1}\left\{\frac{3}{8}w^2 - \frac{1}{6}w^6\right\}\frac{dy}{(1-|y|^2)^{1/2}}.\end{aligned}$$

By the support properties of w, this is finite for $\delta > 0$. The main properties of $\tilde{E}$ are:

Lemma: For $\delta \geq 0$, $0 < s_1 < s_2 < \log 1/\delta$,

i) $\tilde{E}(w(s_2)) - \tilde{E}(w(s_1)) = \int_{s_1}^{s_2} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \, dy \, ds$, so $\tilde{E} \uparrow$.

ii) $\frac{1}{2} \int_{B_1} [\partial_s w \cdot w - 2w^2] \frac{dy}{(1-|y|^2)^{1/2}} = -\int_{s_1}^{s_2} \tilde{E}(w(s)) ds +$
$\frac{1}{3} \int_{s_1}^{s_2} \int_{B_1} \frac{|w|^6}{(1-|y|^2)^{1/2}} \, ds \, dy +$
$\int_{s_1}^{s_2} \int_{B_1} \left\{ (\partial_s w)^2 + \partial_s w \, y \cdot \nabla w + \frac{\partial_s w w |y|^2}{(1-|y|^2)} \right\} \frac{dy}{(1-|y|^2)^{1/2}}$.

iii) $\lim_{s \to \log 1/\delta} \tilde{E}(w(s)) = E$, so that $\tilde{E}(w(s)) \leq E$, $0 < s < \log 1/\delta$.

The proof is computational. Our first improvement is:

Lemma:

$$\int_0^1 \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)} \, dy \, ds \leq C \log 1/\delta.$$

For the proof, we notice that

$$-2\int \frac{(\partial_s w)^2}{(1-|y|^2)} = \frac{d}{ds}\int \left[\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(|\nabla w|^2 - (y\cdot\nabla w)^2) + \frac{3}{8}w^2 - \frac{1}{6}w^6\right]$$
$$\log 1/1-|y|^2 dy$$
$$+ \int \log[(1-|y|^2)+2]\, y\cdot\nabla w\, \partial_s w - \log(1-|y|^2)(\partial_s w)^2 - 2(\partial_s w)^2$$

We then integrate between 0 and 1 and drop the next to last term by sign. The proof is concluded by Cauchy-Schwartz, using the support property of $w(-;\delta)$.

Corollary:

a) $\int_0^1 \int_{B_1} \frac{w^6}{(1-|y|^2)^{1/2}} \leq C(\log 1/\delta)^{1/2}$

b) $\tilde{E}(w(1)) \geq -C(\log 1/\delta)^{1/2}$

The first statement follows from ii), iii) above, Cauchy-Schwartz and the previous lemma. Note that we obtain the power 1/2 on the right hand side, by ignoring by sign the third term on the right of ii) and Cauchy-Schwartz. Part b) follows from i) and $\int_0^1 \tilde{E}(w(s))ds \geq -C(\log 1/\delta)^{1/2}$, which is a consequence of the definition of $\tilde{E}$ and a).

The second improvement is:

Lemma:

$$\int_1^{\log 1/\delta} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \leq C(\log 1/\delta)^{1/2}.$$

This follows from i), iii) and b).

Corollary: There exists $\bar{s}_\delta \in (1, (\log 1/\delta)^{3/4})$ such that

$$\int_{\bar{s}_\delta}^{\bar{s}_\delta + (\log 1/\delta)^{1/8}} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \leq C/(\log 1/\delta)^{1/8}.$$

Proof: Pigeonhole argument; split $(1,(\log 1/\delta)^{3/4})$ into disjoint intervals of length $(\log 1/\delta)^{1/8}$. Their number is $(\log 1/\delta)^{5/8}$, and $\frac{5}{8}-\frac{1}{8}=\frac{1}{2}$.

Note that the length of the s integral tends to infinity, while the bound goes to 0. It is easy to see that if $\bar{s}_\delta \in (1,(\log 1/\delta)^{3/4})$, $\bar{s}_\delta = -\log(1+\delta-\bar{t}_\delta)$, then

$$\left|\frac{(1-\bar{t}_\delta)}{1+\delta-\bar{t}_\delta}-1\right| \leq C\delta^{1/4},$$

which goes to 0 with δ. From this, and the compactness of $\bar{K}$, we can find $\delta_j \to 0$, so that $w(y,\bar{s}_{\delta_j}+s;\delta_j)$ converges, for $s\in[0,S]$ to $w^*(y,s)$ in $C([0,S];H_0^1\times L^2)$ and w^* solves our self-similar equation in $B_1\times[0,S]$. By the above corollary, w^* is independent of s. Also, if $w^*\equiv 0$, it is not difficult to see that $(0,0)\in\bar{K}$, which contradicts our assumptions, since then u is globally defined. Thus, $w^*\in H_0^1(B_1)$ solves the (degenerate) elliptic equation

$$\frac{1}{\rho}\operatorname{div}\left(\rho\nabla w^* - \rho(y\cdot\nabla w^*)y\right) - \frac{3}{4}w^* + w^{*5} = 0,$$

$\rho(y) = (1-|y|^2)^{-1/2}$ and is non-zero. We next point out that w^* satisfies the additional (crucial) estimates:

$$\int_{B_1} \frac{|w|^6}{(1-|y|^2)^{1/2}} + \int_{B_1} \frac{|\nabla w^*|^2 - (y\cdot\nabla w^*)^2}{(1-|y|^2)^{1/2}} < \infty.$$

Indeed, for the first estimate, it suffices to show that, uniformly in j large, we have

$$\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S} \int_{B_1} \frac{|w(y,s;\delta_j)|^6}{(1-|y|^2)^{1/2}}\, dy\, ds \le C$$

which follows from ii) and the choices of $\bar{s}_{\delta_j}$, by i) and iii). The proof of the second estimate follows from the first one, iii) and the formula for $\tilde{E}$.

The conclusion of the proof is obtained by showing that a w^* in $H^1_0(B_1)$, solving the degenerate elliptic equation, with the additional bounds must be zero. This follows from a unique continuation argument. Recall that, for $|y| < 1 - \eta_0$, $\eta_0 > 0$, the linear operator is uniformly elliptic, with smooth coefficients, and the nonlinearity is critical. An old argument of Trudinger's shows that w^* is bounded on $\{|y| \leq 1 - \eta_0\}$, for each $\eta_0 > 0$. Thus, if we show that $w^* \equiv 0$ near $|y| = 1$, the standard Carleman unique continuation principle will show that $w^* \equiv 0$.

Near $|y| = 1$, it is easy to see that our equation is modeled (in tangential variables $z \in \mathbb{R}^2$ and normal ones $r \in \mathbb{R}$, $r > 0$, and we are looking near $r = 0$) by

$$r^{1/2}\, \partial_r (r^{1/2}\, \partial_r\, w^*) + \Delta_z\, w^* + c\, w^* + (w^*)^5 = 0.$$

Our information on w^* translates into $w^* \in H^1_0([0,1] \times \{|z| < 1\})$. Our crucial additional estimates are:

$$\int_0^1 \int_{|z|<1} |w^*(r,z)|^6 \frac{dz}{r^{1/2}}\, dr + \int_0^1 \int_{|z|<1} |\nabla_z w^*(r,z)|^2 \frac{dz}{r^{1/2}}\, dr < \infty.$$

To conclude, we exploit the degeneracy of the equation. We "desingularize" the problem by letting $r = a^2$, setting $v(a,z) = w^*(a^2,z)$, so that $\partial_a v(a,z) = 2r^{1/2}\, \partial_r w^*(r,z)$. Our equation becomes

$$\partial_a^2 v + \Delta_z v + cv + v^5 = 0, \quad 0 < a < 1,\ |z| < 1,\ v\,|_{a=0} = 0.$$

Our bounds become:

$$\int_0^1 \int_{|z|<1} |\nabla_z v(a,z)|^2\, dz\, da = \int_0^1 \int_{|z|} |\nabla_z w^*(r,z)|^2 \frac{dz}{r^{1/2}}\, dr < \infty$$

and

$$\int_0^1 \int_{|z|<1} |\partial_a v(a,z)|^2 \frac{da}{a} dz = \int_0^1 \int_{|z|<1} |\partial_r w^*(r,z)|^2 \, dr \, dz < \infty,$$

so that $v \in H_0^1((0,1] \times B_1)$, and in addition, $\partial_a v(a,z) \mid_{a=0} = 0$. We extend v by 0 to $a < 0$ and see that the extension is an H^1 solution to the same equation. By Trudinger's arguments, it is bounded. But it vanishes for $a < 0$, and thus, by Carleman's unique continuation theorem, $v \equiv 0$. Hence, $w^* \equiv 0$, giving our contradiction.

We conclude with an apparently different version of the theorem, which turns out to be equivalent to it.

Theorem: $(u_0, u_1) \in \dot{H}^1 \times L^2$. Assume that $E(u_0, u_1) < E(W, 0)$.

- **i)** If $\int |\nabla u_0|^2 + |u_1|^2 < \int |\nabla W|^2$, $I = \mathbb{R}$, $\|u\|_{S(\mathbb{R})} < \infty$.
- **ii)** If $\int |\nabla u_0|^2 + |u_1|^2 > \int |\nabla W|^2$, then $-\infty < T_- < T_+ < \infty$.
- **iii)** $\nexists\, (u_0, u_1)$ with $\int |\nabla u_0|^2 + |u_1|^2 = \int |\nabla W|^2$.

The Theorem follows from the following two variational claims.

Claim 1: If $E(u_0, u_1) < E(W, 0)$, $\|\nabla u_0\|^2 < \|\nabla W\|^2$ if and only if $\|\nabla u_0\|^2 + \|u_1\|^2 < \|\nabla W\|^2$.

Claim 2: If $E(u_0, u_1) < E(W, 0)$, $\|\nabla u_0\|^2 > \|\nabla W\|^2$ if and only if $\|u_1\|^2 + \|\nabla u_0\|^2 > \|\nabla W\|^2$.

Definition: Let u be a solution of (NLW), $T_+ < \infty$. We say that u is a type II blow-up solution if $\sup_{0<t<T_+} \|\nabla u(t)\| + \|\partial_t u(t)\| < \infty$. We say that u is a type I blow up solution if $\lim_{t+T_+} = \infty$. A priori the two notions are not mutually exclusive (mixed asymptotics).

Definition: $x_0 \in \mathbb{R}^3$, u a type II blow-up solution. We say that x_0 is regular if $\forall\, \varepsilon > 0$, $\exists\, R > 0$ such that, $\forall\, t \in [0, T_+)$,

$$\int_{|x-x_0|<R} |\nabla_{x,t}\, u(t)|^2 + \frac{|u(t)|^2}{|x-x_0|^2} \leq \varepsilon.$$

If x_0 is not regular we say that it is singular, and call S the set of singular points.

Theorem: u type II blow-up solution. Then $\exists N \in \mathbb{N}\backslash\{0\}$ and N distinct points $m_1, \dots, m_N$ of $\mathbb{R}^3$, with $S = \{m_1, \dots, m_N\}$. Moreover, there exists $(v_0, v_1) \in \dot{H}^1 \times L^2$ such that

a) $(\vec{u}(t)) \rightharpoonup (v_0, v_1)$ weakly in $\dot{H}^1 \times L^2$

b) $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi \equiv 1$ near each singular point, then

$$(*) \qquad \lim_{t \to T_+} \|(1-\varphi)[\vec{u}(t) - (v_0, v_1)]\|_{\dot{H}^1 \times L^2} = 0.$$

Definition: Let u be as in Theorem. Let v be the solution of (NLW) such that $(v(T_+), \partial_t v(T_+)) = (v_0, v_1)$. We call v the regular part of u at T_+ and $a = u - v$ the singular part. Note that $(*)$ and finite speed of propagation implies that $\operatorname{supp} \vec{a}(t) \subset \bigcup_{k=1}^{N} \{(x,t) : |x - m_k| \leq |T_+ - t|\}$.

Sketch of the proof of Thm. $(T_+ = 1)$: The main point is the following

Lemma: i) If for $x_0 \in \mathbb{R}^3$, $t_0 \in (0,1)$, $R > 0$ we have

$$\int_{|x-x_0| \leq |t_0-1|+R} |\nabla_{x,t}\, u(t_0)|^2 + \frac{|u(t_0)|^2}{|x-x_0|^2} \leq \delta_1,$$

δ_1 small, and $\varphi \in C_0^\infty(\mathbb{R}^3)$, supp $\varphi \subset \{|x-x_0| < R\}$, then $(\varphi u(t), \varphi \partial_t\, u(t))$ has a limit in $\dot{H}^1 \times L^2$ as $t \longrightarrow 1$.

ii) If $t_0 \in (0,1)$, $R > 0$ and

$$\int_{|x|>R} |\nabla_{x,t}\, u(t_0)|^2 + \frac{|u(t_0)|^2}{|x-x_0|^2} \leq \delta_1, \text{supp}\, \varphi \subset \{|x| > R + |1-t_0|\}$$

then $(\varphi u(t), \varphi \partial_t\, u(t))$ has a limit in $\dot{H}^1 \times L^2$ as $t \longrightarrow 1$.

The Lemma is a simple consequence of the local theory of the Cauchy problem and finite speed of propagation.

Corollary: If m is a singular point, $t \in I_{\max}$, then

$$\delta_1 \leq \int_{|x-m|\leq|t-1|} |\nabla_{x,t}\, u(t)|^2 + \frac{|u(t)|^2}{|x-m|^2}.$$

If the estimate fails, $\exists\, m \in S$, $t_0 \in I$, $\varepsilon > 0$ such that

$$\int_{|x-m|\leq|t_0-1|+\varepsilon} |\nabla_{x,t}\, u(t_0)|^2 + \frac{|u(t_0)|^2}{|x-m|^2} < \delta_1.$$

Choose $\varphi \in C_0^\infty$, $\varphi \equiv 0$ if $|x-m| \geq \varepsilon$, $\varphi \equiv 1$ if $|x-m| \leq \varepsilon/2$. Then, $\varphi \cdot \vec{u}(t)$ converges in $\dot{H}^1 \times L^2$ as $t \to 1$, which contradicts m being a singular point. Note that the Corollary and type II give S finite. Once we have this, the Lemma gives the Theorem.

In preparation for the proof of the radial soliton resolution we will prove a few crucial linear facts.

We consider solutions v of (LW), and define $E_L(\vec{v}(t)) = \frac{1}{2}\int |\nabla_{x,t} v(t)|^2$, which is constant in t. We let $e_L(\vec{v})(x,t) = \frac{1}{2}|\nabla_{x,t} v(t)|^2$, and recall that for sufficiently smooth solutions, $\partial_t e_L(\vec{v}) = \sum_{j=1}^{3} \partial_{x_j}(\partial_{x_j} v(x,t) \cdot \partial_t v(x,t))$, from which energy conservation follows. We next introduce, for $A \geq 0$, $E^{\text{out}}_{L,A}(\vec{v}(t)) = \frac{1}{2}\int_{|x| \geq |t|+A} |\nabla_{x,t} v(t)|^2$. This is (for all $A \geq 0$) a non-increasing function of t, for $t > 0$, non-decreasing for $t < 0$. To see this, we just integrate $\partial_t e_L$ for $|x| \geq t + A$, $t > 0$ and $0 < s_0 < t < t_0$, to obtain

$$\frac{1}{2}\int_{|x|>t_0+A} |\nabla_{x,t} v(t_0)|^2 - \frac{1}{2}\int_{|x|>s_0+A} |\nabla_{x,t} v(s_0)|^2 = \frac{-1}{2\sqrt{2}} \int_{\substack{|x|=t+A \\ s_0<t<t_0}} \left|\nabla v + \frac{x}{|x|}\cdot v_t\right|^2 d\sigma.$$

Let

$$E^{\text{out}}_{L,A,+\infty} = \lim_{t\uparrow\infty} E^{\text{out}}_{L,A}(t), \quad E^{\text{out}}_{L,A,-\infty} = \lim_{t\downarrow-\infty} E^{\text{out}}_{L,A}(t).$$

Proposition: (DKM, 09) Let v be a radial solution of (LW) in $\mathbb{R}^3$, $A \geq 0$. Then, for all $t \geq 0$ or for all $t \leq 0$,

$$\int_{r>|t|+A} |\partial r(r\,v(t))|^2 + r^2|\partial_t v(r,t)|^2\,dr \geq \frac{1}{2}\int_{r\geq A}\left[\partial_r(r\,v_0(r))^2 + r^2\,v_1(r)\right]ds.$$

Proof: Let $f(r,t) = r\,v(r,t)$, extended oddly for $r < 0$. Let $f_0 = r\,v_0(r)$, $f_1 = r\,v_1(r)$. Then $(\partial_t^2 - \partial_r^2)f = 0$, $t \in \mathbb{R}$, $r \in \mathbb{R}$. Also, since $v(t) \in \dot{H}^1(\mathbb{R}^3)$, Hardy's inequality gives $\int_0^\infty |v(r,t)|^2\,dr < \infty$, and so $f_0 \in \dot{H}^1(\mathbb{R})$, $f_1 \in L^2(\mathbb{R})$. Let

$$z_1(r,t) = \frac{\partial_r f(r,t) + \partial_t f(r,t)}{2},\; z_2(r,t) = \frac{\partial_r fr(t) - \partial_t f(r,t)}{2}.$$

Then, $z_1^2(r,0) + z_2^2(r,0) = \frac{(\partial_r f_0)^2 + f_1^2}{2}$. Hence, for at leat one i,

$$\int_{r\geq A} z_i^2(r,0)\,dr \geq \frac{1}{4}\int_{r\geq A}\left[(\partial_r f_0)^2 + f_1^2\right]dr.$$

Say it is $i = 1$.

Note that $(\partial_t - \partial_r) z_1(r,t) = 0$. Hence, for $\tau < r$ we have $\partial_\tau z_1(r-\tau, t+\tau) = 0$. Then (because $i = 1$) fix $t < 0$. Then,

$$\begin{aligned} \int_{r \geq A} z_1^2(r,0)\,dr &= \int_{r\geq A} z_1^2(r-t,t)\,dr = \int_{r \geq A+|t|} z_1^2(r,t)\,dr \\ &\leq \frac{1}{2} \int_{r \geq A+|t|} \left[\partial_r (r\,v(t))^2 + r^2 (\partial_t v(r,t))^2\right] dr \end{aligned}$$

since $\frac{(a+b)^2}{4} \leq \frac{1}{2}(a^2+b^2)$.

We will use the following calculation often:

$$\int_A^\infty (\partial_r (r\,h(r)))^2\,dr = \int_A^\infty [h(r) + r\,\partial_r\,h(r)]^2\,dr$$

$$= \int_A^\infty (\partial_r\,h(r))^2 r^2\,dr + \int_A^\infty \partial_r (r\,h^2(r))\,dr$$

$$= \int_A^\infty [\partial_r\,h(r)]^2 r^2\,dr - Ah^2(A) \leq \int_A^\infty [\partial_r\,h(r)]^2 r^2\,dr.$$

Cor: For $t > 0$ or for $t < 0$,

$$E_{L,A}^{\text{out}}(t) \geq \frac{1}{2}\int_{r>A} \left[\frac{\partial_r (r\,v_0)^2}{2} + \frac{r^2\,v_1^2}{2}\right] dr.$$

Remark: Since for $\vec{v}(0) \in C_0^\infty$, $|v(x,t)| \leq C/t$ (dispersive estimate) the Corollary does not loose information. However, for $A > 0$, $\int_{r\geq A}\left[\frac{(\partial_r(r\,v_0))^2}{2}+\frac{r^2\,v_1^2}{2}\right]dr$ cannot be replaced by $\int_{r\geq A}\frac{r^2(\partial_r\,v_0)^2}{2}+\frac{r^2\,v_1^2}{2}\,dr$, as can be seen by choosing $v_0 = \begin{cases} 1/r & r > A \\ 1/A & 0 < r \leq A \end{cases}$ $v_1 = 0$. The corresponding $v(r,t) = 1/r$ for $r > t+A$, by finite speed of propagation, and hence $E^{\text{out}}_{L,A,+\infty} = 0$, but $\int_{r\geq A}\frac{r^2(\partial_r\,v_0)^2}{2}\,dr$ is not 0. Note however that $\int_{r\geq A}(\partial_r(r\,v_0))^2dr = 0$.

The next ingredient holds for odd dimensions, as a consequence of the strong Huygens principle. Later on we shall see a proof that holds also in even dimensions.

Lemma: v a solution of (LW) in $\mathbb{R}^3$, $\{\lambda_n\}$, $\{t_n\}$ real, $\lambda_n > 0$, $\lim_{n\to\infty} |\lambda_n / t_n| = \infty$. Let $v_n(x,t) = \frac{1}{\lambda_n^{1/2}} v\left(\frac{x}{\lambda_n}, \frac{t}{\lambda_n}\right)$. Then,

$$\lim_{R\to\infty} \overline{\lim_{n}} \int_{||x|-|t_n||>R\,\lambda_n} |\nabla_{x,t} v_n(t_n)|^2 + \frac{|v_n(t_n)|^2}{|x|^2}\, dx = 0.$$

Sketch of proof: By scaling we can assume $\lambda_n \equiv 1$. We then approximate v by the solution $\tilde{v}$ corresponding to data $(\tilde{v}_0, \tilde{v}_1)$ supported in B_R. By the strong Huygens principle, if $|t_n| > R$, $\tilde{v}(t_n)$ is supported on $||x| - |t_n|| \le R$ and the result follows.

The meaning of the result is that for large time, the energy concentrates on the boundary of the light-cone.

Remark: Note that the following "dispersive property" of non-zero solution v of (LW) holds: $\exists\, R>0,\ \eta>0$ such that, for all $t \geq 0$ or for all $t \leq 0$,

$$\int_{|x|>R+|t|} |\nabla_{x,t}\, v(t)|^2\, dx \geq \eta > 0.$$

Indeed, since $\int |\nabla v_0|^2 + v_1^2 \neq 0$, and equals $\int_0^\infty [\partial_r (r\, v_0)]^2 + (r\, v_1)^2\, dr \neq 0$, we can find $R>0,\ \eta>0$ such that

$$\int_R^\infty [\partial_r (r\, v_0)]^2 + (r\, v_1)^2\, dr \geq 2\eta > 0,$$

and the claim follows from the Proposition. The key idea for the proof of soliton resolution of (NLW) in the radial case is that this holds also for (NLW), except for scaling of W. Our first task is to show this.

We introduce the following notation: let $(u_0, u_1) \in \dot{H}^1 \times L^2$, radial, $R > 0$. Then, $(\tilde{u}_0, \tilde{u}_1) = \Psi_R(u_0, u_1)$ is given by

$$\tilde{u}_0(r) = \begin{cases} u_0(r) & \text{if } r > R \\ u_0(R) & \text{if } 0 < r \leq R \end{cases} \qquad \tilde{u}_1(r) = \begin{cases} u_1(r) & \text{if } r > R \\ 0 & \text{if } r < R \end{cases}$$

Note that $\|(\tilde{u}_0, \tilde{u}_1)\|^2_{\dot{H}^1 \times L^2} = \int_{|x|>R} |\nabla u_0|^2 + u_1^2$.

Proposition 1: Let u be a global in time, radial solution of (NLW) such that, for some $R > 0$,

$$\lim_{t \to +\infty} \int_{|x|>R+t} |\nabla_{x,t}\, u(t)|^2 = \lim_{t \to -\infty} \int_{|x|>R+|t|} |\nabla_{x,t}\, u(t)|^2 = 0.$$

Then, either (u_0, u_1) is compactly supported or $\exists \lambda > 0$, $i \in \{\pm 1\}$ such that $(u_0, u_1) - \left(\frac{i}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right), 0 \right)$ is compactly supported.

To prove this we use two Lemmas.

Lemma 1: u as in Proposition. Let $v(r,t) = r\,u(r,t)$, $v_0 = r\,u_0$, $u_1 = r\,u_1$. Then, there exists $C_0 > 0$ such that, if for some $r_0 > 0$ we have

$$\int_{r_0}^{\infty} [(\partial_r u_0)^2 + u_1^2] r^2 \, dr \leq \delta_0,$$

where δ_0 is small, then

$$\int_{r_0}^{\infty} [(\partial_r v_0)^2 + v_1^2] \, dr \leq C_0 \frac{|v_0(r_0)|^{10}}{r_0^5}.$$

Furthermore, for r, r', $r_0 \leq r, r' \leq 2r_0$, we have

$$|v_0(r) - v_0(r')| \leq \sqrt{C_0} \frac{|v_0(r)|^5}{r^2} \leq \sqrt{C_0}\, \delta_0^2 |v_0(r)|.$$

Sketch of proof: The second statement follows from the first one by the fundamental theorem and the fact that, for $r \geq r_0$

$$\frac{1}{r}\, v_0^2(r) = r\, u_0^2(r) \leq \int_r^{\infty} [\partial_s\, u_0(s)]^2\, s^2\, ds \leq \delta_0.$$

To prove the first inequality let u_L be the solution of (LW), $v_L = r\, u_L$. By the Corollary, for all $t \geq 0$, or all $t \leq 0$

$$\int_{r_0+|t|}^{\infty} \left[(\partial_r\, u_L(t))^2 + (\partial_t\, u_L(t))^2\right] r^2\, dr \geq \frac{1}{2} \int_{r_0}^{\infty} (\partial_r\, v_0)^2 + v_1^2.$$

Let $(\tilde{u}_0, \tilde{u}_1) = \Psi_{r_0}(u_0, u_1)$, $\tilde{u}_L$ the corresponding solution of (LW).

For δ_0 small, the local theory of the Cauchy problem gives

$$\begin{aligned}
\|(\vec{\tilde{u}} - \vec{\tilde{u}}_L\,(t)\|_{\dot{H}^1\times L^2} &\leq C\|(\tilde{u}_0,\tilde{u}_1)\|^5_{\dot{H}^1\times L^2} \\
&= C\left[\int_{r_0}^{\infty}\{[\partial_r\,u_0]^2 + u_1^2\}\,r^2\,dr\right]^{5/2} \\
&= C\left[\int_{r_0}^{\infty}[\partial_r\,v_0]^2 + v_1^2\,dr + r_0\,u_0^2(r_0)\right]^{5/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{r_0+|t|}^{\infty}\left([\partial_r\,\tilde{u}_L(t)]^2 + [\partial_t\,\tilde{u}_L(t)]^2\right)r^2dr &\leq 2\int_{r_0+|t|}^{\infty}\left([\partial_r\,\tilde{u}(t)]^2 + [\partial_t\,\tilde{u}(t)]^2\right)r^2\,dr \\
&+ C\left[\int_{r_0}^{\infty}[(\partial_r\,v_0)^2 + v_1^2]\,dr + r_0\,u_0^2(r_0)\right]^5
\end{aligned}$$

Since, by finite speed $\vec{u}(r,t) = \vec{\tilde{u}}(r,t)$, $\vec{u}_L(r,t) = \vec{\tilde{u}}_L(r,t)$, for $r > r_0 + |t|$ we obtain, for all $t \geq 0$ or all $t \leq 0$,

$$\begin{aligned} \int_0^\infty [(\partial_r v_0)^2 + v_1^2]\, dr \quad &\leq \quad 4\int_{r_0+|t|}^\infty \left([\partial_r u(t)]^2 + [\partial_t u(t)]^2\right) r^2 dr \\ &+ \quad C\left[\int_{r_0}^\infty [\partial_r v_0]^2 + v_1^2\, dr + r_0\, u_0^2(r_0)\right]^5 . \end{aligned}$$

Letting $t \to \pm\infty$ according to which holds above, and since $\int_{r_0}^\infty (\partial_r v_0)^2 + v_1^2 \leq \int_{r_0}^\infty [(\partial_r u_0)^2 + u_1^2]\, r^2 dr \leq \delta_0$, for δ_0 small, we get the bound, since $r_0^5\, u_0^{10}(r_0) = v_0^{10}(r_0)/r_0^5$.

Lemma 2: $v_0(r)$ has a limit $\ell \in \mathbb{R}$ as $r \to \infty$. Furthermore, $\exists\, C > 0$ such that, $\forall\, r \geq 1$

$$|v_0(r) - \ell| \leq C/r^2.$$

Sketch of proof: We first give a preliminary bound: $|v_0(r)| \leq C \cdot r^{1/10}$. Indeed,

$$|v_0(2^{n+1}r_0)| \leq |v_0(2^{n+1}r_0) - v_0(2^n r_0)| + |v_0(2^n r_0)| \leq \left[1 + \sqrt{C_0}\,\delta_0^2\right] |v_0(2^n r_0)|.$$

If δ_0 is small, $\left(1 + \sqrt{C_0}\,\delta_0^2\right) \leq 2^{1/10}$ and we iterate. Next,

$$\lim_{r\to\infty} v_0(r) = \ell : |v_0(2^n r_0) - v_0(2^{n+1}r_0)| \leq \sqrt{C_0}\,\frac{|v_0(2^n r_0)|^5}{(2^n r_0)^2},$$

so that $|v_0(2^n r_0) - v_0(2^{n+1}r_0)| \leq C/[2^n]^{2-5/10} = C/2^{3n/2}$, so that $\sum_{n\geq 0} |v_0(2^n r_0) - v_0(2^{n+1}r_0)| < \infty$, so that $\lim_{n\to\infty} v_0(2^n r_0) = \ell$.

Using $|v_0(r)| \leq Cr^{1/10}$, and the difference estimate, we conclude that $\lim_{r\to\infty} v_0(r) = \ell$, so that v_0 is bounded, and so, $|v_0(2^{n+1}r) - v_0(2^n r)| \leq \frac{C}{(2^n r)^2}$, which gives the estimate for $|v_0(r) - \ell|$.

To prove the Proposition 1, we distinguish $\ell = 0$, $\ell \neq 0$. If $\ell = 0$ we show that (v_0, v_1) is compactly supported. Indeed, by the smallness of δ_0,

$$|v_0(2^{n+1}r) - v_0(2^n r)| \leq \sqrt{C_0}\,\delta_0^2 |v_0(2^n r)| \leq \frac{1}{4}|v_0(2^n r)|,$$

so that $|v_0(2^{n+1}r)| \geq \frac{3}{4}|v_0(2^n r)|$ and iterating, $|v_0(r)| \leq \left(\frac{4}{3}\right)^n |v_0(2^n r)|$. Since $\ell = 0$, Lemma 2 gives $|v_0(2^n r)| \leq \frac{C}{2^{2n} r^2} = \frac{C}{4^n r^2}$, and so $|v_0(r)| \leq 3^{-n}/r^2$, which shows $v_0(r) \equiv 0$, $r > r_0$. But $\int_r^\infty [\partial_s v_0(s)^2 + v_1^2(s)]\, ds \leq C_0 \frac{|v_0(r)|^{10}}{r^5}$ by Lemma 1, and so v_1 also has compact support.

If $\ell \neq 0$, we show that $\exists\, \lambda > 0$ and a $\pm$ sign such that $\left(u_0 \pm \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right), u_1\right)$ has compact support. Recall $W(r) = \frac{1}{(1+r^2/3)^{1/2}}$. Then, for large r, $\left|\frac{1}{\lambda^{1/2}} W\left(\frac{r}{\lambda}\right) - \frac{\sqrt{3}\,\lambda^{1/2}}{r}\right| \leq C/r^3$. Hence, by Lemma 2, $\exists\, C > 0$ such that

$$\left|\pm \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right) - u_0(r)\right| \leq C/r^3,$$

where $\lambda = \ell^2/3$ and the sign is the one of ℓ. Rescaling u and possibly replacing it by $-u$, we can assume that $|u_0(r) - W(r)| \leq C/r^3$, $r \geq 1$. We then let $h = u - W$, $H = rh$ and show for R_0 large, $r_0 > R_0$, we have

$$\int_{r_0}^{\infty} \left[(\partial_r H_0) + H_1^2\right] dr \leq \frac{1}{16} \frac{H_0^2(r_0)}{r_0}.$$

From this we can easily conclude that $(H_0(r), H_1(r)) = (0,0)$ for large r, by a similar argument to the one used before.

To show the claim, we note that $h = u - W$ verifies the equation $\partial_t^2 h - \Delta h = (h+W)^5 - W^5$. We use the linearized equation and truncate data and the potential, which is now

$$V = \begin{cases} W(x) & \text{if } |x| > R_0 + |t| \\ W(R_0 + |t|) & \text{if } |x| \le R_0 + |t| \end{cases}$$

for which we can obtain a local theory of the Cauchy problem for R_0 large. We then argue similarly to the case $\ell = 0$. This gives the Proposition 1.

Proposition 2. Let u be a non-zero radial solution of (NLW) such that $\forall\,\lambda > 0$, all $\pm$ signs, $\left(u_0 \pm \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right), u_1\right)$ is not compactly supported. Then, there exists $R > 0$, $\eta > 0$ and $\tilde{u}$ a globally defined solution of (NLW) such that $\tilde{u}$ scatters in both time directions, and for all $t \ge 0$ or all $t \le 0$,

$$\int_{|x|>R+|t|} |\nabla_{x,t}\,\tilde{u}(x,t)|^2 \ge \eta, \quad u(x,t) = \tilde{u}(x,t),\ |x| > R + |t|.$$

Sketch of proof. If (u_0, u_1) is not compactly supported, $(\tilde{u}_0, \tilde{u}_1) = \psi_R(u_0, u_1)$, where R is so large that $0 < \|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1 \times L^2} \leq \eta_0$, η_0 given by the local theory of the Cauchy problem. By a simple argument, the conclusion is verified for $\tilde{u}$ unless

$$\lim_{t \uparrow \infty} \int_{|x|>R+|t|} |\nabla_{x,t}\, \tilde{u}(t)|^2 = \lim_{t \downarrow -\infty} \int_{|x|>R+|t|} |\nabla_{x,t}\, \tilde{u}(t)|^2 = 0.$$

But then, by Proposition 1, $(\tilde{u}_0, \tilde{u}_1)$ is of compact support (excluded) or $(\tilde{u}_0, \tilde{u}_1) - \left(\frac{i}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right), 0\right)$ has compact support (excluded). Thus, let (u_0, u_1) be of compact support, not $(0,0)$. Let

$$\rho(u_0, u_1) = \inf\{r > 0 : |\{s > r : (u_0(s), u_1(s)) \neq (0,0)\}| = 0\}$$

where $0 < \rho(u_0, u_1) < \infty$. Pick $0 < R < \rho(u_0, u_1)$, $(\tilde{u}_0, \tilde{u}_1) = \Psi_R(u_0, u_1)$, R so that $0 < \|(\tilde{u}_1, \tilde{u}_2)\|_{\dot{H}^1 \times L^2} < \eta_0$. The channel of energy for $\tilde{u}_L$ finishes the proof.

Proposition 3. Let $R_0 > 0$ be large, to be chosen. Let u be a radial solution of (NLW) such that $(h_0, h_1) = (u_0 \pm W, u_1)$ is compactly supported and not $\equiv 0$. Then,

a) $\exists$ a solution $\breve{u}$ of (NLW), defined for $t \in [-R_0, R_0]$ and $R' \in (0, \rho(h_0, h_1))$ such that $(\breve{u}_0(r), \breve{u}_1(r)) = (u_0(r), u_1(r))$ for $r > R'$ and $\forall\, t \in [0, R_0]$ or $\forall\, t \in [-R_0, 0]$

$$\rho(\breve{u}(t) \pm W, \partial_t \breve{u}(t)) = \rho(h_0, h_1) + |t|.$$

b) Assume further that $\rho(h_0, h_1) > R_0$. Let $R < \rho(h_0, h_1)$ be close to $\rho(h_0, h_1)$. Then, $\exists\, \eta > 0$ and a global radial solution $\tilde{u}$, which scatters, such that $(\tilde{u}_0(r), \tilde{u}_1(r)) = (u_0(r), u_1(r))$ for $r > R$ and, for all $t \geq 0$ or all $t \leq 0$,

$$\int_{|x|>R+|t|} |\nabla_{x,t}\, \tilde{u}(t)|^2 \geq \eta.$$

Sketch of proof. For b) we use the argument in the proof of Proposition 2 when (u_0, u_1) is compactly supported, using instead of (NLW) the equation for h, truncating W as in Proposition 1, for R_0 large.

For a) we use a linearization around W. Up to a sign change, $(u_0, u_1) = (W, 0) + (h_0, h_1)$, $0 \leq \rho(h_0, h_1) < \infty$. Since W is globally defined, by the Perturbation Theorem, for any U with $\|(W, 0) - (U_0, U_1)\|_{\dot{H}^1 \times L^2} < \varepsilon$, ε small, $[-R_0, R_0] \subset I_{\max}(U)$. Choose R' close to $\rho(h_0, h_1)$, smaller, so that $0 < \|(\breve{h}_0, \breve{h}_1)\|_{\dot{H}^1 \times L^2} < \varepsilon$, $(\breve{h}_0, \breve{h}_1) = \psi_{R'}(h_0, h_1)$. Let now $\breve{u}$ be the solution of (NLW) with initial data $(W + \breve{h}_0, \breve{h}_1)$. By the definition of ε, $\breve{u}$ and $\breve{h} = u - W$ are defined on $[-R_0, R_0]$. By finite speed of propagation, $(\breve{u}, \partial_t \breve{u}) = (W, 0)$ for $r > \rho(h_0, h_1) + |t|$, $t \in [-R_0, R_0]$. Thus,

$$\rho(\breve{h}(t), \partial_t \breve{h}(t)) \leq \rho(h_0, h_1) + |t|, \quad t \in [-R_0, R_0].$$

We will show that, for all $t \in [-R_0, 0]$, or for all $t \in [0, R_0]$,

$$(+) \qquad \rho\left(\breve{h}(t), \partial_t \breve{h}(t)\right) = \rho(h_0, h_1) + |t|.$$

Recall that

$$\begin{cases} \partial_t^2 \breve{h} - \Delta \breve{h} = (W + \breve{h})^5 - W^5 \\ (\breve{h}, \partial_t \breve{h}) \mid_{t=0} = (\breve{h}_0, \breve{h}_1). \end{cases}$$

For t_0 small the equation has a good local theory of the Cauchy problem and the solution for small data is close to the solution of (LW). Then, if $R' < \rho_0 < \rho(h_0, h_1)$, $(g_0, g_1) = \psi_{\rho_0}(\breve{h}_0, \breve{h}_1)$, for t_0 small we have,

$$\sup_{|t| \leq t_0} \|\vec{g}(t) - \vec{g}_L(t)\|_{\dot{H}^1 \times L^2} \leq \frac{1}{10} \|(g_0, g_1)\|_{\dot{H}^1 \times L^2}.$$

For all $t \in [-t_0, 0]$, or for all $t \in [0, t_0]$,

$$\int_{|x| > \rho_0 + |t|} |\nabla_{x,t} g_L(t)|^2 \geq \frac{1}{2} \int_{|x| > \rho_0} |\nabla g_0|^2 + g_1^2 - \frac{1}{2} \rho_0 g_0^2(\rho_0).$$

If ρ_0 is close enough to $\rho(h_0, h_1)$, $\rho_0 g_0^2(\rho_0) \leq \frac{1}{4} \|\nabla g_0\|^2_{L^2(r > r_0)}$.

Hence $\forall\, t \geq 0$ or $\forall\, t \leq 0$, $t \in [-t_0, t_0]$,

$$\int_{|x| \geq \rho_0 + |t|} |\nabla_{x,t}\, g(t)|^2 \geq \frac{1}{40} \int_{|x| \geq \rho_0} |\nabla g_0|^2 + g_1^2.$$

By finite speed, we can replace g by $\breve{h}$ in the left hand side. Hence $\rho(\breve{h}(t), \partial_t \breve{h}(t)) \geq \rho_0 + |t|$, $\forall\, t \in [-t_0, 0]$ or $\forall\, t \in [0, t_0]$. Letting $\rho_0 \to \rho$, we see that

$$\rho(\breve{h}(t), \partial_t \breve{h}(t)) = \rho(h_0, h_1) + |t|\,, \quad t \in [-t_0, 0] \text{ or } t \in [0, t_0].$$

An iteration concludes the proof.

Remark: The meaning of Propositions 2, 3 is that after we wait for a while, we have the dispersive property, unless one is $\pm W$, as desired.

We now give a Corollary of Propositions 1, 2, which expresses them in terms of profiles, which is how we will apply them here.

Lemma on profiles: Consider a radial non-zero profile (the index j is irrelevant)

$$U^j_{L,n}(x,t) = \frac{1}{\lambda_{jn}^{1/2}} U^j_L\left(\frac{x}{\lambda_{jn}}, \frac{t-t_{jn}}{\lambda_{jn}}\right).$$

Assume that $\lim_n(-t_{jn}/\lambda_{jn}) = \pm\infty$, or $t_{jn} \equiv 0$. Assume one of the following holds:

a) For all $\mu > 0$, signs $\pm$, $\left(U^j_0 \pm \frac{1}{\mu^{1/2}} W\left(\frac{\cdot}{\mu}\right), U^j_1\right)$ is not compactly supported, or

b) $\exists$ a sign $\pm$ such that $(U^j_0 \pm W, U^j_1)$ is compactly supported and $\rho(U^j_0 \pm W, U^j_1) > R_0$.

Then there exists $\tilde{U}^j_L$ a solution of (LW) and a sequence $\{\rho_{jn}\}_n$, $n > 0$, such that the nonlinear profile $\tilde{U}^j$ associated to $\tilde{U}^j_L$ and $\{\lambda_{jn}, t_{jn}\}$ is globally defined and scatters, $\vec{\tilde{U}}^j_{L,n}(x,0) = \vec{U}^j_{L,n}(x,0)$, for $|x| \geq \rho_{jn}$ and $\exists\, \eta_j > 0$ such that for all $t \geq 0$ or all $t \leq 0$, $\forall\, n$

$$\int_{|x|\geq \rho_{jn}+|t|} \left|\nabla_{x,t}\, \tilde{U}^j_n(t)\right|^2 dx \geq \eta_j > 0.$$

When $t_{jn} \equiv 0$, this follows directly from Propositions 1, 2. The other cases are simpler.

We now turn to the soliton resolution in the radial case.

Theorem. (DKM) Let u be a radial solution of (NLW). Then, one of the following holds:

a) Type I blow-up: $T_+ < \infty$ and

$$\lim_{t \uparrow T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} = \infty$$

b) Type II blow-up: $T_+ < \infty$, $\exists\,(v_0, v_1) \in \dot{H}^1 \times L^2$, $J \in \mathbb{N} \backslash \{0\}$ and $\forall j \in \{1, \ldots, J\}$, $i_j \in \{\pm 1\}$, $\lambda_j(t) > 0$,

$$0 < \lambda_1(t) \ll \ldots \ll \lambda_{J(t)} \ll (T_+ - t),$$

$$(u(t), \partial_t u(t)) = \left(\sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{1/2}} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + (v_0, v_1) + o(1).$$

c) $T_+ = +\infty$ and $\exists$ a solution v_L of (LW), $J \in \mathbb{N}$, $\forall j \in \{1, \ldots, J\}$, $i_j \in \{\pm 1\}$, $\lambda_j(t) > 0$, $0 < \lambda_1(t) \ll \ldots \ll \lambda_J(t) \ll t$ and

$$(u(t), \partial_t u(t)) = \left(\sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{1/2}} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + (v_L(t)\, \partial_t v_L(t)) + o(1)$$

in $\dot{H}^1 \times L^2$.

We will sketch the proof of c). The first step is to show that $\exists \{t_n\} \uparrow \infty$ such that $\{\vec{u}(t_n)\}$ is bounded in $\dot{H}^1 \times L^2$.

Proposition. Let u be a radial solution of (NLW), $T_+ = \infty$. Then $E(u_0, u_1) \geq 0$ and

$$\underline{\lim}_{t \uparrow +\infty} \|\nabla_{x,t}\, u(t)\|^2 \leq 3\, E(u_0, u_1).$$

The proof is by contradiction, using the blow-up argument of Levine type.

The next step is the extraction of the free wave (radiation term) in c). We use the Proposition above.

Lemma: Let u be a radial solution of (NLW), $T_+ = +\infty$. Then $\exists\, v_L$ such that, $\forall A \in \mathbb{R}$

$$\lim_{t\uparrow\infty} \int_{|x|\geq t-A} |\nabla_{x,t}(v_L - u)(t)|^2 dx = 0.$$

The meaning of this is that at any finite distance from the boundary of the light cone, any global radial solution of (NLW) behaves like a linear wave.

The next ingredient for the proof of c) is used to make explicit how the dispersive estimates (Lemma on profiles) give the decomposition. This is what we call the "channel of energy" argument.

Inductive Lemma: Let u be a globally defined radial solution of (NLW). There exists no sequence $\{t_n\}$ tending to infinity, with the following properties:

$\exists$ a bounded sequence $\{(u_{0n}, u_{1n})\}_n$ in $\dot{H}^1 \times L^2$ and a sequence $\{\rho_n\}_n$, $\rho_n \geq 0$, such that

$$|x| \geq \rho_n \Rightarrow (u(x,t_n), \partial_t u(x,t_n)) = (u_{0n}(x), u_{1n}(x)),$$

$\exists J_0 \in \mathbb{N}$, $i_1, \ldots, i_{J_0} \in \{\pm 1\}$ such that (u_{0n}, u_{1n}) has a profile decomposition of the form

$$\begin{aligned}(u_{0n}, u_{1n}) &= \vec{v}_L(t_n) + \sum_{j=1}^{J_0} \left(\frac{i_j}{\lambda_{jn}^{1/2}} W\left(\frac{x}{\lambda_{jn}}\right), 0 \right) \\ &+ \sum_{j=J_{0+1}}^{J} \vec{U}^j_{L^n}(0) + (w^J_{0n}, w^J_{1n})\end{aligned}$$

where, for all $j \geq J_{0+1}$, the nonlinear profile U^j is globally defined and scatters in both time directions and $\exists\, \varepsilon_0 > 0$ such that one of the following holds:

a) $\exists j_0 \geq J_0 + 1$ such that $\forall\, t \geq 0$ or $\forall\, t \leq 0$

$$\int_{|x| \geq \rho_n + |t|} |\nabla_{x,t}\, U_n^{j_0}(t)|^2 \geq \varepsilon_0$$

or

b) For at least one sign $\pm$

$$\lim_{J \to \infty} \varliminf_{n} \inf_{\pm\, t \geq 0} \int_{|x| \geq \rho_n + |t|} |\nabla_{x,t}\, w_n^J(t)|^2 \geq \varepsilon_0.$$

The proof is by contradiction, assuming the existence of a $\{t_n\}$. After extraction we have $\lim_{n \to \infty} \rho_n / t_n = L \in [0, +\infty]$ and it is not hard to show that $L \leq 1$. Moreover, it is also not hard to show, using that W is not of compact support, that for any $j \in \{1, \dots, J_0\}$, $\lim_{n \to \infty} \frac{\lambda_{jn}}{t_n} = 0$.

Let now v be the solution of (NLW) such that $\lim_{t\uparrow+\infty} \|\vec{v}(t) - \vec{v}_L(t)\|_{\dot{H}^1\times L^2} = 0$.
Translating time if necessary, we can assume that v is defined for $t \in [0, \infty)$.
The statement is proved by induction on J_0.

Case $J_0 = 0$. Let u_n be the solution with data (u_{0n}, u_{1n}). By the approximation theorem u_n is defined on $[-t_n, \infty)$ for large n and

$$\vec{u}_n(x,t) = \vec{v}(x, t_n + t) + \sum_{j=1}^{J} \vec{U}_n^j(x,t) + w_n^J(x,t) + r_n^J(x,t),$$

where $\lim_J \overline{\lim_n} \sup_{[-t_n,\infty)} \|\vec{r}_n^J(t)\|_{\dot{H}^1\times L^2} = 0$.

Assume first that a) holds for $t \geq 0$ or b) holds with $+$. Then, by orthogonality, for n large, $t \geq 0$,

$$\int_{|x|\geq \rho_n + t} |\nabla_{x,t}(u_n(t) - v(t+t_n))|^2 \geq \varepsilon_0/2.$$

By finite speed and the fact that for $|x| \geq \rho_n$ we have $\vec{u}(x, t_n) = (u_{0n}, u_{1n})$, for large n, $t \geq 0$, we have

$$\int_{|x|\geq \rho_n + t} |\nabla_{x,t}(u(x, t_n + t) - v(x, t + t_n))|^2 \geq \varepsilon_0/2,$$

so that

$$\varliminf_{t\uparrow\infty} \int_{|x|>t+(\rho_n - t_n)} |\nabla_{x,t}(u - v)(t)|^2 > 0,$$

which contradicts the construction of v_L.

Next, assume that a) holds for all $t \leq 0$ or b) holds with $-$.

Choose now $t = -t_n$ in the expansion, and use orthogonality again. Then, for large n,

$$\int_{|x|\geq \rho_n + t_n} |\nabla_{x,t}(u_n(x, -t_n) - v(x, 0))|^2 \geq \varepsilon_0/2.$$

But then, by finite speed $u_n(x,-t_n) = u_0(x)$, so

$$\int_{|x|\geq \rho_n + t_n} |\nabla(u_0 - v(0))|^2 + |u_1 - \partial_t v(0)| \geq \varepsilon_0/2.$$

This is a contradiction because $t_n \to \infty$.

Inductive step: Fix $J_1 \geq 0$, assume that the Lemma holds for $J_0 \leq J_1$. Let $t_n \uparrow \infty$ be such that the assumptions of the Lemma are satisfied with $J_0 = J_1 + 1$. We then use a similar argument, but where we replace "the first" W with $(\tilde{U}_0^1, \tilde{U}_1^1) = \Psi_T(W, 0)$, with T large. The argument is then similar to the previous case, adjusting the time sequence.

The main step in the proof of c).

Proposition. Let $t_n \to \infty$ be such that $\{\vec{u}(t_n)\}$ is bounded on $\dot{H}^1 \times L^2$, v_L the radiation term. Then (after extraction) $\exists\, J \geq 0$, $i_1, \ldots, i_J \in \{\pm 1\}$, $0 < \lambda_{1n} \ll \ldots \ll \lambda_{Jn} \ll t_n$ such that

$$\vec{u}(t_n) - \vec{v}_L(t_n) - \sum_{j=1}^{J} \left(\frac{i_j}{\lambda_{jn}^{1/2}} W(x/\lambda_{jn}), 0 \right) \underset{n}{\longrightarrow} (0,0) \quad \text{in } \dot{H}^1 \times L^2.$$

Sketch of proof. If not, after extraction, $\vec{u}(t_n)$ has a profile decomposition of the form

$$\vec{u}(t_n) = \vec{v}_L(t_n) + \sum_{j=1}^{J_0} \left(\frac{i_j}{\lambda_{jn}^{1/2}} W\left(\frac{x}{\lambda_{jn}}\right), 0 \right) + \sum_{j=J_0+1}^{J} \vec{U}_{Ln}^j(0) + (w_{0n}^J, w_{1n}^J),$$

where $J_0 \geq 0$, $i_j \in \{\pm 1\}$ and for $j \geq J_0 + 1$ one of the following holds: $\lim_{n\to\infty} -t_{jn}/\lambda_{jn} \in \{-\infty, +\infty\}$ or $t_{jn} \equiv 0$ and $\forall\, \lambda > 0$, $(U_{L0}^j, U_{L1}^j) \neq \left(\pm \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right), 0 \right)$.

Furthermore, one of the following holds: $U_L^{J_0+1} \not\equiv 0$, or $\forall j \geq J_0 + 1$, $U_L^j \equiv 0$ and $\varliminf_n \|(w_{0n}^J, w_{1n}^J)\|_{\dot{H}^1 \times L^2} > 0$. Note that $\forall j \in \{1, \dots, J_0\}$ $\lim_n \frac{\lambda_{jn}}{t_n} = 0$, by the fact that ∇W is not of compact support. Then, to carry out the proof, we split it into various cases. In each case, using the "Lemma on profiles", we reduce to the situation where $\vec{u}(t_n)$ coincides for $|x| \geq \rho_n$, for some $\rho_n \geq 0$, with a sum of rescaled $\pm W$ and of globally defined profiles creating energy channels in the cone $|x| \geq \rho_n + |t|$, for $t \geq 0$ or $t \leq 0$. This is due to our "dispersive property". We then reach a contradiction from the "Inductive Lemma", which constitutes the channel of energy argument. This can be performed directly along $\{t_n\}$, unless the profile $\vec{U}_L^j(0)$, $j \geq J_0 + 1$ which is "further from 0" is of the form $(\pm W + h_0^j, h_1^j)$ where $\rho(h_0^j, h_1^j) \leq R_0$. In this case, we use case a) of Proposition 3 (growth of support in time), finite speed of propagation and the approximation theorem, to get to the previous situation along another sequence $\{\tilde{t}_n\}$. The details are lengthy.

Proof of c) for all times (sketch). Let u be a global solution. We know that there exists $\{t_n\}_n \to \infty$, such that $\{\vec{u}(t_n)\}$ is bounded and a radiation term v_L. By the Proposition, $\exists J \in \mathbb{N}$, $(i_1, \ldots, i_J) \in \{\pm 1\}^J$, $\{\lambda_{jn}\}_n$, $0 < \lambda_{1n} \ll \ldots \ll \lambda_{Jn} \ll t_n$, such that (after extraction)

$$\lim_{n\to\infty} \left\| \vec{u}(t_n) - \vec{v}_L(t_n) - \sum_{j=1}^{J} \left(\frac{i_j}{\lambda_{jn}^{1/2}} W(x/\lambda_{jn}), 0 \right) \right\|_{\dot{H}^1 \times L^2} = 0.$$

Step 1: Convergence of norms. We first show that

$$\lim_{t\uparrow\infty} \|\nabla(u - v_L)(t)\|^2 = J\|\nabla W\|^2$$

$$\lim_{t\uparrow\infty} \|\partial_t(u - v_L)(t)\|^2 = 0$$

Note that this proves the boundedness for all times, and fixes the number of bubbles. To show that $\lim_{t\uparrow\infty} \|\nabla(u - v_L)(t)\|^2 = J\|\nabla W\|^2$, we assume not.

Recalling that, along $\{t_n\}$ we have the right limit, if this fails, by continuity of the flow and the intermediate value theorem, for $|\varepsilon| \neq 0$, but small, we can find $\{t_n^1\}_n \to \infty$ such that $\lim_{t\to\infty} \|\nabla(u - v_L)(t_n^1)\|^2 = J\|\nabla W\|^2 + \varepsilon$. Hence $\nabla u(t_n^1)$ is bounded in L^2, and thus so is $u(t_n^1)$ in L^6. By preservation of energy, $\{\partial_t u(t_n^1)\}$ is bounded in L^2. Thus, by the Proposition, for some $J^1 \in \mathbb{N}$, $\lim_{t\to\infty} \|\nabla(u - v_L)(t_n^1)\|^2 = J^1\|\nabla W\|^2$. But, choosing ε appropriately, we reach a contradiction. A similar argument shows that $\lim_{t\uparrow\infty} \|\partial_t(u - v_L)(t)\|^2 = 0$.

Step 2: Choice of the scaling. For $j = 1, \ldots, J, t > 0$, large,

$$B_j = (j-1)\|\nabla W\|^2 + \int_{|x|\leq 1} |\nabla W|^2,$$

$$\lambda_j(t) = \inf\left\{\lambda > 0 : \int_{|x|\leq\lambda} |\nabla(u - v_L)(t)|^2 \geq B_j\right\}.$$

Claim: If $\theta_n \to \infty$, after extraction, $\exists\,(i_1^1, \ldots, i_J^1) \in \{\pm 1\}^J$ such that

$$\lim_{n\to\infty} \left\| \vec{u}(\theta_n) - \vec{v}_L(\theta_n) - \sum_{j=1}^{J} \frac{i_j^1}{\lambda_j^{1/2}(\theta n)} \left(W\left(\frac{x}{\lambda_j(\theta_n)}\right), 0\right) \right\|_{\dot{H}^1\times L^2}^2 = 0,$$

and $\lambda_1(\theta_n) \ll \lambda_2(\theta_n) \ll \ldots \ll \lambda_J(\theta_n) \ll \theta_n$.

By the Proposition, the claim holds with $\{\lambda_{jn}^1\}_n$. Let $j \in \{1, \ldots, J\}$. If $r_0 > 0$, we have

$$\lim_{n\to\infty} \int_{|x|\leq r_0\,\lambda_{jn}^1} |\nabla(u - v_L)(\theta_n)|^2 = (j-1)\|\nabla W\|^2 + \int_{|x|\leq r_0} |\nabla W|^2.$$

If $r_0 < 1$, $r_0\,\lambda_{jn}^1 < \lambda_j(\theta_n)$ for large n. If $r_0 > 1$, $r_0\,\lambda_{jn}^1 > \lambda_j(\theta_n)$ for large n. Hence $\lim_{n\to\infty} \frac{\lambda_j(\theta_n)}{\lambda_{jn}^1} = 1$. This gives our claim. Note that $\lambda_j(t)$ is continuous in t.

Step 3: Choice of signs and end of proof. Let $\mathcal{I} = (\alpha_1, \dots, \alpha_J) \in \{\pm 1\}^J$. Let

$$\mathcal{A}_{\mathcal{I},\delta} = \Bigg\{ f \in \dot{H}^1 : \exists\, \lambda_1, \dots, \lambda_J > 0 \text{ such that}$$
$$\left\| f - \sum_{j=1}^{J} \frac{\alpha_j}{\lambda_j^{1/2}} W(x/\lambda_j) \right\|_{\dot{H}^1} + \sum_{j=1}^{J-1} \frac{\lambda_j}{\lambda_{j+1}} < \delta \Bigg\}.$$

Then, it is not hard to see that $\exists\, \delta_0 > 0$ small such that, if $\mathcal{I} \neq \mathcal{I}^1$, $f \in \mathcal{A}_{\mathcal{I},\delta_0}$, $g \in \mathcal{A}_{\mathcal{I}',\delta_0} \Rightarrow \|f-g\|_{\dot{H}^1} > \delta_0$. Choose such a δ_0. By Step 2, $\exists\, t_0 > 0$ such that, $\forall\, t > t_0$, $u(t) - v_L(t) \in \bigcup_{\mathcal{I} \in \{\pm 1\}^J} \mathcal{A}_{\mathcal{I},\delta_0}$. But then, by the continuity of the flow in $\dot{H}^1$, $\exists\, \mathcal{I}$ such that $\forall\, t > t_0$, $u(t) - v_L(t) \in \mathcal{A}_{\mathcal{I},\delta_0}$. Letting $\mathcal{I} = (i_1, \dots, i_J)$, using Step 2 and the fact above, we see that

$$\lim_{t \uparrow \infty} \left\| \vec{u}(t) - \vec{v}_L(t) - \sum_{j=1}^{J} \left(\frac{i_j}{\lambda_j(t)^{1/2}} W(x/\lambda_j(t)), 0 \right) \right\|_{\dot{H}^1 \times L^2} = 0.$$

This concludes the proof.